FORMAL LANGUAGES AND IDEMPOTENT SEMIGROUPS

Helena Maria Sezinando

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I declare that the accompanying thesis has been composed by myself and that it is a record of my own work. No part of the thesis has been accepted in any previous application for a higher degree.

Helena Maria da Encarnação Sezinando

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John M. Howie

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ABSTRACT

The structure of the lattice **LB** of varieties of idempotent semigroups or bands (as universal algebras) was determined by Birjukov, Fennemore and Gerhard. Wismath determined the structure of a related lattice: the lattice **LBM** of varieties of band monoids. In the first two parts we study several questions about these varieties.

In Part I we compute the cardinalities of the Green classes of the free objects in each variety of **LB** [**LBM**]. These cardinalities constitute a useful piece of information in the study of several questions about these varieties and some of the conclusions obtained here are used in parts II and III.

Part II concerns expansions of bands [band monoids]. More precisely, we compute here the cut-down to generators of the Rhodes expansions of the free objects in the varieties of **LB**. We define Rhodes expansion of a monoid, its cut-down to generators and we compute the cut-down to generators of the Rhodes expansions of the free objects in the varieties of **LBM**.

In Part III we deal with Eilenberg varieties of band monoids. The last chapter is particularly concerned with the description of the varieties of languages corresponding to these varieties.

CONTENTS

INTRODUCTION

We recall that an idempotent semigroup (band) is a semigroup in which every element is idempotent. The bands have been studied from different aspects, concerning for example, construction, classification, characterization of different subclasses, the lattice of subsemigroups of a band, congruences, varieties of bands, etc. This thesis is concerned with the lattice **LB** of all varieties of bands (as universal algebras), the lattice **LBM** of all varieties of band monoids and the lattice **LFBM** of all Eilenberg varieties of finite band monoids (sometimes called pseudovarieties of band monoids). Three main questions are studied.

The first question concerns the computation of the cardinalities of the Green classes of the free objects in each variety of bands and in each variety of band monoids. This study (carried out in chapter 3) belongs to the classical theory of semigroups and some of the information obtained is used later in the study of the two other questions.

We analyse first the varieties of rectangular bands. In the remaining varieties of bands the $\mathcal{J}$-classes of the free objects are defined by an invariant, called the *content* (the content of a word being the set of variables occurring in it). This allows us to use a common technique for the study of $\mathbf{LB_0}$, that is, the lattice obtained from **LB** by taking away the rectangular bands. We introduce a distinction between the "left" and "right" parts of $\mathbf{LB_0}$: the *left varieties of* $\mathbf{LB_0}$ are the varieties $\mathbf{V}(xy = yx)$, $\mathbf{V}(ax = axa)$, $\mathbf{V}(\overline{R}_2 = \overline{Q}_2)$, $\mathbf{V}(R_n = T_n)$, $\mathbf{V}(\overline{R}_m = \overline{T}_m)$, $T \in \{Q, S\}$, $n, m \geq 3$, n odd, m even, and the variety **B** of all bands. The right varieties are defined dually. Let $F_A(V)$ be the free object in the variety V, generated by A. We show that for every proper variety V the number of $\mathcal{L}$-classes in each $\mathcal{J}$-class J of a free object $F_A(V)$ in $V \in \mathbf{LB_0}$ is equal to the number of $\mathcal{L}$-classes in the corresponding $\mathcal{J}$-class in $F_A(V_r)$, where V_r is the maximum of the set of right varieties contained in V. Since the cardinality of an $\mathcal{R}$-class of a $\mathcal{J}$-class is just the number of $\mathcal{L}$-classes in that $\mathcal{J}$-class, we obtain in this way the cardinality of an arbitrary $\mathcal{R}$-class of J. We have dual results for $\mathcal{L}$-classes. It follows from this that the cardinality of J is the product of the number of $\mathcal{L}$-classes of the corresponding $\mathcal{J}$-class of $F_A(V_r)$ by the number of $\mathcal{R}$-classes of the corresponding $\mathcal{J}$-class of $F_A(V_l)$, where V_l is the maximum of the set of left varieties contained in V.

We also analyse the following question: given two varieties of bands V and W such that $V \subseteq W$, for which cardinalities of the finite alphabet A do the free objects in V and W generated by A coincide?

We deduce that two free objects (generated by A) in $V(R_l = T_l)$ and $V(R_{l'} = T_{l'}), T \in \{Q, S\}, l, l' \geq 3, l \neq l'$, coincide if and only if $l, l' >| A |$. More generally, if V and W are varieties of bands such that $V \subseteq W$, then $F_A(V)$ and $F_A(W)$ coincide if and only if the number of $\mathcal{R}$-classes of the $\mathcal{J}$-classes J_A and J'_A of $F_A(V_l)$ and $F_A(W_l)$, respectively, coincide and the number of $\mathcal{L}$-classes of the $\mathcal{J}$-classes J''_A and J'''_A of $F_A(V_r)$ and $F_A(W_r)$, respectively, also coincide, where J_A, J'_A, J''_A and J'''_A are the $\mathcal{J}$-classes formed by the words of content A.

Finally, in this first part we prove that the cardinalities of the Green classes of the free objects in a variety V of bands and in the corresponding variety $V \cap \mathrm{Mon}$ of band monoids (where Mon is the set of all monoids) coincide.

The classical theory of semigroups (or local theory of semigroups) 'aims at classifying semigroups up to isomorphism'. (We quote Rhodes.) Recently, a new theory was developed, the *global theory of semigroups*, which incorporates the classical results. It 'tries to classify semigroups up to some reasonable equivalence relation (like, e.g., homotopy-type in topology) to get a good feel for what an arbitrary semigroup looks like, not more classifications up to isomorphism in special cases'. (We quote Rhodes again.) In this newer theory the notion of *expansion*, introduced by Rhodes in 1969, plays a fundamental role. The second question studied in this thesis concerns Rhodes expansions of bands and band monoids. We study the cut-down to generators A, $\hat{(.)}^{\mathcal{L}}_{A}$, of the Rhodes expansion $\hat{(.)}^{\mathcal{L}}$ of the free objects in the varieties of bands.

We follow the strategy used in the first part: We show first that the cut-down to generators of the Rhodes expansion (with respect to $\mathcal{L}$) of the free object in a rectangular band is isomorphic to that free object. Next, we work on $\mathbf{LB_0}$ and show that the cut-down to generators A of the Rhodes expansion of a free object $F_A(V)$ in a variety V of $\mathbf{LB_0}$ is isomorphic to $F_A(V^r)$, where V^r is the minimum of the set of *strictly right* varieties of $\mathbf{LB_0}$ containing V. By a strictly right variety of $\mathbf{LB_0}$ we mean a proper right variety of $\mathbf{LB_0}$ different from $\mathbf{V}(xy = yx), \mathbf{V}(R_2 = Q_2)$.

Also, we define the *Rhodes expansion of a monoid*, $\hat{(.)}^{\mathcal{L}}_{e}$, and its *cut-down to generators* A, $\hat{(.)}^{\mathcal{L}}_{e,A}$, in such a way that the functors $(.)^{I} \circ \hat{(.)}^{\mathcal{L}}_{A}$ and $\hat{(.)}^{\mathcal{L}}_{e,A} \circ (.)^{I}$ are naturally equivalent. This leads to the fact that the cut-down to generators A of the Rhodes expansion of the free object $F_A(V)$ in a non trivial variety V of band monoids is also isomorphic to $F_A(V^r)$. (V^r is defined for band monoids in a way similar to the one done for bands.)

Finally, the analysis made leads us to the conclusion that the only varieties of band monoids closed for Rhodes expansions are the trivial variety and the variety of all band monoids. This is not true for varieties of bands but also holds for Eilenberg varieties of band monoids. (Notice that the Rhodes expansion of a finite semigroup is finite.)

The theory of languages, which in the beginning was developed independently, became strictly connected with the theory of semigroups, when Eilenberg, in 1976, introduced the notions of *Eilenberg variety of semigroups* [*monoids*], a certain class of finite semigroups [monoids], and *variety of languages*, a certain class of recognizable languages. In his Variety Theorem he established a bijection between these two classes. This gave rise to an enormous amount of questions. Between them one arises naturally: given a certain variety of semigroups [monoids] what is the corresponding variety of languages, by Eilenberg's theorem? (and vice-versa).

The third question broached in this thesis concerns the description of the varieties of languages corresponding to the Eilenberg varieties of finite band monoids.

A variety of languages is a function defined by the image of each alphabet. This image is a certain boolean algebra. We denote by $A^*(R_n = S_n)$, $n \geq 2$, the image of the alphabet A by the variety of languages corresponding to the variety of band monoids $\mathbf{VM}(R_n = S_n)$. We show that for $n \geq 1$, $A^*(R_{n+1} = S_{n+1})$ is the boolean algebra generated by the languages L of *type* L_p, $1 \leq p \leq n$, where

$$L = A_{p-1}^* a_{p-1} \dots A_1^* a_1 A_0^* a_2 A_2^* \dots a_p A_p^*$$

if p is even,

$$L = A_p^* a_p \dots A_1^* a_1 A_0^* a_2 A_2^* \dots a_{p-1} A_{p-1}^*$$

if p is odd, and in both cases $A_0 \subset A_1 \subset \dots \subset A_p \subseteq A$, $a_i \in A_i \setminus A_{i-1}$, $i = 1, \dots p$.

We also show that if $|A| = N < n$ then the boolean algebras $A^*(R_{n+1} = S_{n+1})$ and $A^*(R_{N+1} = S_{N+1})$ coincide.

More precisely, our work in this thesis is organized in the following way:

The first part consists of three chapters. The first two are introductory chapters. Chapter 1 contains the classical definitions and results on the theory of categories and on the theory of semigroups, which will be used throughout this thesis. In this chapter we also present the category of A-generated semigroups, which will be consistently used in the second part. Chapter 2 contains classical material on Birkhoff varieties, specified for semigroups. We also present here the notion of Eilenberg variety of semigroups. Section two of this chapter is more specific. We present here the family of invariants used by Gerhard and Petrich for solving the word problem for varieties of bands and develope some of their results on these invariants. This chapter prepares the next one, which, as we have mentioned, is dedicated to the study of the Green classes of the free objects in the varieties of bands [band monoids].

The second part consists of chapters 4 and 5. In chapter 4 we present the definitions of Rhodes expansion of a semigroup and of its cut-down to generators. After presenting some properties of these expansions, we define Rhodes expansion of a monoid and its cut-down to generators, establishing a relantionship between these notions and the corresponding ones for semigroups. Chapter 5 is dedicated to the computation of the cut-downs to generators of the Rhodes expansions of the free objects in each variety of bands [band monoids].

The last part is exclusively dedicated to finite band monoids. In chapter 6 we present the notion of variety of languages and some general results concerning the varieties of languages corresponding to the Eilenberg varieties of finite band monoids, depending on the cardinality of the alphabet. This chapter prepares the last one, where after presenting the family of generators, constructed by Gerhard, for the Eilenberg varieties of band monoids and doing some computations on these generators, we describe the varieties of languages corresponding to the varieties of finite band monoids.

PART I – VARIETIES OF BANDS

CHAPTER 1. SEMIGROUPS

We summarize here a great number of definitions and simple results concerning the general theory of semigroups.

In the first section we give some notions on the theory of categories, since we will have several occasions to use the language of Category Theory.

Sections 2 and 4 consist mainly of the classic definitions and results on the category of semigroups.

In section 3 another category is introduced, namely the category of A-generated semigroups, which will be consistently used in chapters 4 and 5.

Finally, section 5 contains the main results on idempotent semigroups or bands that we will need throughout the thesis.

A full discussion concerning the material presented in sections 2, 4 and 5 can be found in the books by Clifford and Preston **[6]**, Howie **[16]** and Petrich **[18]**.

1. Categories

Here we give a concise list of the definitions which will be used (without reference).

Definition 1.1.1. A *category* **C** is given by the data (i)–(iv), having the properties (v), (vi).

(i) A class called the *class of objects of* **C**.

(ii) A rule that associates to each pair of objects X, Y of **C** a set $\mathbf{Mor}_{\mathbf{C}}(X, Y)$ called the *set of morphisms from* X *to* Y. Its elements are called *the morphisms from* X *to* Y.

(iii) A rule that associates to each triple X, Y, Z of objects of $\mathbf{C}$ a mapping

$$\mathrm{Mor}_{\mathbf{C}}(X,Y) \times \mathrm{Mor}_{\mathbf{C}}(Y,Z) \to \mathrm{Mor}_{\mathbf{C}}(X,Z),$$

described by $(g, f) \mapsto f \circ g$.

(iv) A rule that associates to each object X of $\mathbf{C}$ a morphism

$$\mathrm{id}_X \in \mathrm{Mor}_{\mathbf{C}}(X,X).$$

(v) Given objects X, Y of $\mathbf{C}$ and $f \in \mathrm{Mor}_{\mathbf{C}}(X,Y)$

$$f \circ \mathrm{id}_X = \mathrm{id}_Y \circ f = f.$$

(vi) Given objects X, Y, Z, W of $\mathbf{C}$, $f \in \mathrm{Mor}_{\mathbf{C}}(Z,W)$, $g \in \mathrm{Mor}_{\mathbf{C}}(Y,Z)$, $h \in \mathrm{Mor}_{\mathbf{C}}(X,Y)$

$$(f \circ g) \circ h = f \circ (g \circ h).$$

Examples of categories are the category of all semigroups and the category of all monoids, which we present next section. In section 3 we shall present another category — the category of A-generated semigroups.

We now set down some familiar definitions in an arbitrary category.

Definition 1.1.2. A morphism f in the category $\mathbf{C}$ is an *epimorphism* if for every pair g, h of morphisms in $\mathbf{C}$ with $g \circ f = h \circ f$ we must have $g = h$.

The dual of definition 1.1.2 is

Definition 1.1.3. A morphism f in the category $\mathbf{C}$ is a *monomorphism* if for every pair g, h of morphisms in $\mathbf{C}$ with $f \circ g = f \circ h$ we must have $g = h$.

Definition 1.1.4. A morphism $f : A \to B$ in the category $\mathbf{C}$ is an *isomorphism* if there is a morphism $g : B \to A$ such that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. We call such g an *inverse* of f. We say that A and B are *isomorphic*, $A \simeq B$, just in case $Mor_{\mathbf{C}}(A, B)$ contains an isomorphism.

Fact 1.1.5. An isomorphism in a category $\mathbf{C}$ is both a monomorphism and a epimorphism. The converse is not true.

Definition 1.1.6. A *functor* F from a category $\mathbf{C}$ to a category $\mathbf{D}$ is a rule which assigns to every object A of $\mathbf{C}$ an object FA of $\mathbf{D}$, and to every morphism $f : A \to B$ of $\mathbf{C}$ a morphism $Ff : FA \to FB$ of $\mathbf{D}$ such that

(1) $F\mathrm{id}_A = \mathrm{id}_{FA}$, for every object A of $\mathbf{C}$;

(2) If $f \circ g$ is defined in $\mathbf{C}$ then $Ff \circ Fg$ is defined in $\mathbf{D}$ and $Ff \circ Fg = F(f \circ g)$.

The most trivial example of a functor is, for each category $\mathbf{C}$, the identity functor $\mathrm{Id}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ which sends A to A and f to f.

Definition 1.1.7. If $F, G : \mathbf{C} \to \mathbf{D}$ are functors then a *natural transformation* from F to G is a rule that assigns to each object A of $\mathbf{C}$ a morphism $\eta_A : FA \to GA$ of $\mathbf{D}$ in such a way that associated with every morphism $f : A \to B$ in $\mathbf{C}$ there is a commutative diagram

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\eta_A"'] & FB \arrow[d, "\eta_B"] \\ GA \arrow[r, "Gf"] & GB \end{tikzcd}$$

We denote such a natural transformation by $\eta : F \to G$. When this situation obtains, the associated morphisms η_A are said to be *natural*. If each η_A is an isomorphism then we say that η is a *natural isomorphism*.

Definition 1.1.8. We say that two functors $F, G : \mathbf{C} \to \mathbf{D}$ are *naturally equivalent* and write $F \approx G$, if there is a natural isomorphism $\eta : F \to G$.

Remark 1.1.9. To compose two natural transformations $\eta : F \to G$ and $\mu : G \to H$ we simply paste together commutative diagrams:

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ {\scriptstyle \eta_A}\downarrow & & \downarrow{\scriptstyle \eta_B} \\ GA & \xrightarrow{Gf} & GB \\ {\scriptstyle \mu_A}\downarrow & & \downarrow{\scriptstyle \mu_B} \\ HA & \xrightarrow{Hf} & HB \end{array}$$

The commutative outer rectangle then yields a natural transformation $\mu \circ \eta : F \to H$ with $(\mu \circ \eta)_A = \mu_A \circ \eta_A$.

For an extensive discussion of categories we refer to the text of Goldblatt [**15**]. (See also [**4**]).

2. The category of semigroups [monoids]

A *semigroup* is a couple formed from a set S and an internal associative law of composition defined on S. This law shall be denoted in a multiplicative way. Given two semigroups S and T a *semigroup morphism* $\varphi : S \to T$ is a function from S into T such that, for all $x, y \in S$, $\varphi(xy) = \varphi(x)\varphi(y)$.

A *monoid* is a triplet formed from a set M, an internal associative law of composition defined over M and a distinct element of M, denoted by id_M, such that, for every $m \in M$, $\mathrm{id}_M m = m \mathrm{id}_M = m$. Given two monoids M and N, a *monoid morphism* $\varphi : M \to N$ is a function from M into N such that $\varphi(xy) = \varphi(x)\varphi(y)$ for every $x, y \in M$ and such that $\varphi(\mathrm{id}_M) = \mathrm{id}_N$.

The semigroups [monoids], together with the morphisms, form a category, denoted by $\mathcal{S}$ [$\mathcal{M}$].

In $\mathcal{S}$, a morphism is an isomorphism if and only if it is bijective. As usual, we shall identify two isomorphic semigroups.

In $\mathcal{S}$, a morphism is a monomorphism if and only if it is injective.

In $\mathcal{S}$, a surjective morphism is an epimorphism. The converse is not true. (See [17].)

We say that S is a *subsemigroup* of T if there exists an injective morphism $\varphi : S \to T$ and we identify S with $\varphi(S)$.

A *submonoid* of a monoid M is a subsemigroup of M containing id_M.

We say that T is a *quotient* of S if there exists a surjective morphism $\varphi : S \to T$.

We say that a semigroup S *divides* a semigroup T if S is a quotient of a subsemigroup of T. We denote it by $S < T$.

Given a subset A of a semigroup S, there is a smallest subsemigroup of S containing A: it is the intersection of all subsemigroups of S containing A and is denoted by $< A >$ (or $< A >_S$). It consists of all finite products of elements of A. Any subset A of S such that $S =< A >$ is called a *set of generators* of S and S is said to be *generated* by A.

ing A, also denoted by $< A >$ (or $< A >_M$). It is the intersection of all submonoids of M containing A. If $< A >= M$, M is said to be *generated* by A.

(1.2.1.) Given a semigroup S, we denote by S^{I} the monoid obtained from S by the addition of an identity: the support of S^{I} is the disjoint union of S and the singleton $\{1_S\}$ and the law (denoted by $*$) is defined by

$$x * y = xy \quad \text{if } x, y \in S$$
$$1_S * x = x * 1_S = x, \quad \text{for every } x \in S^{\mathrm{I}}.$$

Notice that if S has an identity id_S, then $\mathrm{id}_S \neq 1_S$, that is, S^{I} has a new identity 1_S.

(1.2.2.) Given a semigroup S, we denote by S^1 the following monoid: if S is a monoid, $S^1 = S$; if S is not a monoid, $S^1 = S^{\mathrm{I}}$.

Fact 1.2.3. There is a functor $(.)^{\mathrm{I}}$ from $\mathcal{S}$ to $\mathcal{M}$:

(i) If S is a semigroup, we define

$$(.)^{\mathrm{I}} S = S^{\mathrm{I}}.$$

(ii) If S, T are semigroups and $\varphi : S \to T$ is a morphism, we define

$$(.)^{\mathrm{I}} \varphi = \varphi^{\mathrm{I}}$$

the morphism from S^{I} into T^{I} defined by

$$\varphi^{\mathrm{I}}(s) = \begin{cases} \varphi(s), & \text{if } s \in S; \\ 1_T, & \text{if } s = 1_S; \end{cases}$$

An element e of a semigroup S is *idempotent* if $e^2 = e$. This thesis is exclusively concerned with semigroups whose elements are idempotents.

We call a *zero*, a *right zero* or a *left zero* of a semigroup S, an element, denoted by 0, such that $0s = s0 = 0$, $s0 = 0$ or $0s = 0$ for every $s \in S$, respectively. We call a *zero semigroup*, a *right zero semigroup* or a *left zero semigroup*, a semigroup such that all its elements are zeros, right zeros or left zeros, respectively.

We call an *ideal*, a *right ideal* or a *left ideal* of a semigroup S, a subset I of S such that $S^1IS^1 \subseteq I$, $IS^1 \subseteq I$ or $S^1I \subseteq I$, respectively.

The next concept is of universal algebraic nature.

Definition 1.2.4. If $\{S_i\}_{i\in I}$ is a family of semigroups, their *direct product* is the semigroup defined on the cartesian product $\prod_{i\in I} S_i$ with componentwise multiplication. For a fixed $j \in I$, the mapping π_j from $\prod_{i\in I}$ onto S_j defined by $\pi_j(a_i) = a_j$ is a *projection homomorphism.*

3. The category of A-generated semigroups [monoids]

Definition 1.3.1. Given a set A, let S_A be the following category:

The *objects* of S_A are the pairs (S, f) where S is a semigroup and $f : A \to S$ is a map such that $S =< f(A) >$.

The *morphisms* from (S, f) into (T, g) are those semigroup morphisms $\varphi : S \to T$ such that the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & T \\ S \arrow[ur, "\varphi"'] & \end{tikzcd}$$

We call this category the *category of A-generated semigroups*.

As we shall see in chapter 5, the following facts are important.

Fact 1.3.2. If $(S, f), (T, g)$ are objects of S_A, there is one and only one morphism φ from (S, f) into (T, g).

The morphism φ can be described as follows: if $s = f(a_1) \ldots f(a_n) \in S$, where $a_1, \ldots, a_n \in A$, then

$$\varphi(s) = g(a_1) \ldots g(a_n).$$

Fact 1.3.3. In S_A all morphisms are surjective.

Fact 1.3.4. If φ is a morphism from (S, f) into (T, g) and ψ is a morphism from (T, g) into (S, f), then (S, f) and (T, g) are isomorphic. Indeed, by 1.3.2 we deduce that $\varphi \circ \psi = \mathrm{id}_{(T,g)}$ and $\psi \circ \varphi = \mathrm{id}_{(S,f)}$.

Parallel with the category S_A, we define the *category of A-generated monoids*:

Definition 1.3.5. Given a set A, let $\mathcal{M}_A$ be the following category:

The *objects* of $\mathcal{M}_A$ are the pairs (M, f) where M is a monoid and $f : A \to M$ is a map such that $M =< f(A) >$.

The *morphisms* from (M, f) into (N, g) are those monoid morphisms $\varphi : M \to N$ such that the following diagram is commutative.

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & N \\ M \arrow[ur, "\varphi"'] & \end{tikzcd}$$

We finish this section by establishing a relationship between the two categories presented, by means of a functor. Indeed, we have

Fact 1.3.6. There is a functor $(.)^{\mathrm{I}}$ from $\mathcal{S}_A$ into $\mathcal{M}_A$:

(i) If (S, f) is an object in $\mathcal{S}_A$,

$$(.)^{\mathrm{I}}(S, f) = (S^{\mathrm{I}}, f^{\mathrm{I}})$$

where $f^{\mathrm{I}} : A \to S^{\mathrm{I}}$ is defined by

$$f^{\mathrm{I}}(a) = f(a) \quad (a \in A).$$

(ii) If φ is a morphism from (S, f) into (T, g) (in $\mathcal{S}_A$), define φ^{I} from $(S^{\mathrm{I}}, f^{\mathrm{I}})$ to $(T^{\mathrm{I}}, g^{\mathrm{I}})$ by

$$\varphi^{\mathrm{I}}(s) = \begin{cases} \varphi(s), & \text{if } s \in S; \\ 1_T, & \text{if } s = 1_S. \end{cases}$$

4. Congruences. Green relations

A *congruence* on a semigroup S is an equivalence relation ρ on S compatible on the left and on the right with multiplication, that is, such that, for every $a, b, c \in S$

$$a\rho b \Longrightarrow ac\rho bc, ca\rho cb.$$

The quotient set S/ρ, becomes a semigroup in a natural way if we define

$$[a]_\rho * [b]_\rho = [ab]_\rho \qquad ([a]_\rho\,, [b]_\rho \in S/\rho).$$

The map $\natural_S : S \rightarrow S/\rho$ defined by $\natural(s) = [s]_\rho$ $(s \in S)$ is an epimorphism, called the *canonical epimorphism.*

We now present five equivalence relations of particular importance in the classical theory of semigroups: the *Green's relations*, defined and studied by Green in 1951.

Definition 1.4.1. Let S be a semigroup and let $a, b \in S$. The equivalence relations $\mathcal{R}$, $\mathcal{L}$, $\mathcal{J}$ and $\mathcal{H}$ are defined on S by the following rules:

$$\begin{aligned} a\mathcal{R}b &\Longleftrightarrow aS^1 = bS^1 \\ a\mathcal{L}b &\Longleftrightarrow S^1a = S^1b \\ a\mathcal{J}b &\Longleftrightarrow S^1aS^1 = S^1bS^1 \\ a\mathcal{H}b &\Longleftrightarrow a\mathcal{R}b, a\mathcal{L}b \quad \text{(i.e. } \mathcal{H} = \mathcal{R} \cap \mathcal{L}\text{).} \end{aligned}$$

Fact 1.4.2. The relations $\mathcal{R}$ and $\mathcal{L}$ commute.

Definition 1.4.3. The equivalence relation $\mathcal{D}$ is defined on S by

$$\mathcal{D} = \mathcal{R} \circ \mathcal{L} = \mathcal{L} \circ \mathcal{R}.$$

Notice that $\mathcal{H}, \mathcal{R}, \mathcal{L}, \mathcal{D}, \subseteq \mathcal{J}$.

(1.4.4.) We define also four quasi-order relations (i.e. reflexive and transitive relations) by

$$\begin{aligned} a \leq_{\mathcal{R}} b &\Longleftrightarrow aS^1 \subseteq bS^1 \\ a \leq_{\mathcal{L}} b &\Longleftrightarrow S^1a \subseteq S^1b \\ a \leq_{\mathcal{J}} b &\Longleftrightarrow S^1aS^1 \subseteq S^1bS^1 \\ a \leq_{\mathcal{H}} b &\Longleftrightarrow a \leq_{\mathcal{R}} b, a \leq_{\mathcal{L}} b. \end{aligned}$$

Notation 1.4.5. If $\mathcal{K}$ denotes one of the Green's relations we use the notation $a <_{\mathcal{K}} b$ for $a \leq_{\mathcal{K}} b$ and $(a, b) \notin \mathcal{K}$.

Proposition 1.4.6. *The relations $\leq_{\mathcal{R}}$ and $\mathcal{R}$ are left compatible with multiplication. The relations $\leq_{\mathcal{L}}$ and $\mathcal{L}$ are right compatible with multiplication.*

If R is an $\mathcal{R}$-class and L is an $\mathcal{L}$-class, then $R \cap L \neq \emptyset$ if and only if R and L are within the same $\mathcal{D}$-class. It is usual to represent a $\mathcal{D}$-class by the classical "egg-box" picture below, where each cell represents an $\mathcal{H}$-class, each row an $\mathcal{R}$-class and each column an $\mathcal{L}$-class.

		$\cdots$	
$\vdots$			

5. Bands.

This is a brief survey of concepts and properties related to the title of this section.

A *band* (or *idempotent semigroup*) is a semigroup S in which every element is idempotent.

Proposition 1.5.1. *Let S be a band and let $s, t \in S$. We have*

(i) $s \leq_{\mathcal{L}} t \iff st = s$.

(ii) $s \leq_{\mathcal{R}} t \iff st = t$.

Proposition 1.5.2. *In a band, $\mathcal{D} = \mathcal{J}$.*

A particular case of a band is the one we describe next.

Definition 1.5.3. A *rectangular band* is a pair $(I \times \Lambda, \circ)$ where I and Λ are arbitrary non-empty sets and $\circ$ is defined on the cartesian product $I \times \Lambda$ by

$$(i, \lambda) \circ (j, \mu) = (i, \mu) \qquad (i, j \in I,\ \lambda, \mu \in \Lambda)$$

The following statement illustrates the importance of the rectangular bands.

Proposition 1.5.4. (Clifford 1941; McLean 1952) *A band is a semilattice of rectangular bands, namely its $\mathcal{J}$-classes.*

Notice that 1.5.2 and 1.5.4 yield that in a band each $\mathcal{D}$-class is a rectangular band. We shall consistently use this fact.

We now present the results on rectangular bands we shall use. First we give a characterization of this type of semigroups.

Proposition 1.5.5. *If S is a semigroup then the following statements are equivalent*

(i) *S is a rectangular band.*

(ii) *For all $a, b \in S$, $aba = a$.*

(iii) *For all $a \in S$, $a^2 = a$ and for all $a, b, c \in S$, $abc = ac$.*

Proposition 1.5.6. *Let S be a rectangular band and let $s, t \in S$ be such that $s \leq_{\mathcal{L}} t$. Then $s \,\mathcal{L}\, t$.*

The following statement is a consequence of this result.

Proposition 1.5.7. *Let S be a band and let $s, t \in S$ be such that $s \leq_{\mathcal{L}} t$. Then*

$$s\mathcal{L}t \iff s\mathcal{J}t.$$

Another useful result on rectangular bands is the following:

Proposition 1.5.8. *In a rectangular band there is only one $\mathcal{J}$-class and each $\mathcal{H}$-class has cardinality one.*

Remark 1.5.9. It follows from 1.5.8 that in a given $\mathcal{D}$-class of a band the cardinality of an $\mathcal{L}$ $[\mathcal{R}]$-class is the number of $\mathcal{R}$ $[\mathcal{L}]$-classes inside the $\mathcal{D}$-class.

CHAPTER 2. VARIETIES OF BANDS

This chapter is devoted to varieties and is mainly an introductory chapter.

In the first section we recall some concepts and constructions concerning Birkhoff varieties of semigroups (as varieties of universal algebras).

The second section respects a particular type of varieties of semigroups — the varieties of bands. These are the ones we shall be dealing with in this thesis. This section contains some notation and a large number of results, some of them known, that will be used constantly in the remainder.

Section 3 is a small section where we present the varieties of band monoids.

Finally, the last section is devoted to present a different notion of variety of semigroups [monoids]. It is due to Eilenberg and it concerns finite semigroups [monoids]. We shall work with these varieties, more precisely with Eilenberg varieties of band monoids, in the third part of the thesis.

1. Birkhoff varieties

Even though we will mainly speak about semigroups, the concepts involved in this section are of a universal algebraic nature and we refer to [5] and [7].

Definition 2.1.1. A *variety of semigroups* [*monoids*], in the sense of Birkhoff, is a class of semigroups [monoids] closed under taking subsemigroups [submonoids], quotients and direct products.

This definition is a particular case of the general definition of a variety of algebras.

Examples of varieties of semigroups [monoids] are the class of all semigroups [monoids], denoted by **S** [**M**] and the class of all idempotent semigroups [monoids], denoted by **B** [BM], which we treat in the next section.

The intersection of varieties is again a variety. Hence, since all semigroups [monoids] form a variety, given a class $\mathcal{K}$ of semigroups [monoids] there is a smallest variety containing $\mathcal{K}$. It is called the *variety of semigroups [monoids] generated by* $\mathcal{K}$.

Notation 2.1.2. Let $\mathcal{K}$ be a class of monoids. We denote by $< \mathcal{K} >_{\mathbf{S}}$ and $< \mathcal{K} >_{\mathbf{M}}$ the variety of semigroups and the variety of monoids generated by $\mathcal{K}$, respectively.

There are several possible descriptions of the variety of semigroups [monoids] generated by a certain class of semigroups [monoids]. One of them is the following statement, which is a specification, for semigroups [monoids] of a general theorem on Universal Algebra, due to Tarski:

Proposition 2.1.3. *Let* $\mathcal{K} = \{S_i\}_{i \in I}$ *be a class of semigroups. Then*

$$< \mathcal{K} >_{\mathbf{S}} = \{T \in \mathbf{S} : T < \prod_{j \in J} S_j, J \subseteq I\}.$$

The following result will be useful.

Proposition 2.1.4. *Let* $\mathcal{K}$ *be a class of monoids. Then*

$$<< \mathcal{K} >_{\mathbf{M}} >_{\mathbf{S}} = < \mathcal{K} >_{\mathbf{S}} .$$

Proof. Since $\mathcal{K} \subseteq < \mathcal{K} >_{\mathbf{M}}$, it is clear that $<< \mathcal{K} >_{\mathbf{M}} >_{\mathbf{S}} \supseteq < \mathcal{K} >_{\mathbf{S}}$.

Conversely, let V be a variety of semigroups such that $V \supseteq \mathcal{K}$. We will show that $V \supseteq < \mathcal{K} >_{\mathbf{M}}$.

If $T \in < \mathcal{K} >_{\mathbf{M}}$, then $T < \prod_{i \in I} S_i$ (in $\mathbf{M}$), where the S_i $(i \in I)$ are monoids of $\mathcal{K}$. Since the S_i $(i \in I)$ are in V and T also divides $\prod_{i \in I} S_i$ in $\mathbf{S}$, we deduce that $T \in V$. Hence $< \mathcal{K} >_{\mathbf{M}} \subseteq V$.

We now present the notion of free object in a class of semigroups.

Definition 2.1.5. Let $\mathcal{K}$ be a class of semigroups and A a set. Let (S, f) be an A-generated semigroup such that $S \in \mathcal{K}$.

If for every $T \in \mathcal{K}$ and for every map $\alpha : A \to T$ there is a morphism $\hat{\alpha} : S \to T$ such that the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & S \arrow[dl, "\hat{\alpha}"] \\ T & \end{tikzcd}$$

we say that S has the *universal mapping property* for $\mathcal{K}$ over A. The set $f(A)$ is called a set of *free generators* of S in $\mathcal{K}$ and S is said to be the *free object in* $\mathcal{K}$ *generated by* A.

If it exists, we will denote by $F_A(\mathcal{K})$ the free object in $\mathcal{K}$ generated by A.

Remark 2.1.6. We notice that the morphism $\hat{\alpha}$ of 2.1.5 is unique. Moreover, $\hat{\alpha}$ is surjective if and only if $\alpha(A)$ generates T.

We give next a construction of the free object generated by a set A in the variety of all semigroups.

(2.1.7.) Let A be a non-empty set, called an *alphabet*, whose elements are called *letters.* A *word* in A is a non-empty finite sequence, written $x_1 \dots x_n$, of elements of A. On the set A^+ of all words define a binary operation, called *concatenation* by

$$(x_1 \dots x_m)(y_1 \dots y_n) = x_1 \dots x_m y_1 \dots y_n.$$

This is an associative operation and A is a set of generators of A^+.

Proposition 2.1.8. *The semigroup A^+ has the universal mapping property for* S *over A.*

We call A^+ the *free semigroup on* A.

The *free monoid on* A, denoted by A^*, can be described in similar manner. The only modification required is the inclusion of the *empty word*, denoted by 1. Thus $A^* = A^+ \cup \{1\}$.

Proposition 2.1.8 remains true with "semigroup" replaced by "monoid" and with A^+ replaced by A^*.

In the next propositions we present various useful properties of A^+ and A^*. (See [18].)

Proposition 2.1.9. *Every word $u \in A^+$ has a unique factorization as a product of elements of A.*

Definition 2.1.10. A monoid M is called *equidivisible* if for every $a, b, c, d \in M$, $ab = cd$ implies either $a = cu, ub = d$ for some $u \in M$, or $av = c, b = vd$ for some $v \in M$.

Proposition 2.1.11. *The free monoid A^* is equidivisible.*

One of the most important theorems of Birkhoff says that the varieties are prehis result.

Definition 2.1.12. Let A be a non-empty set and let S be a semigroup. Let $w, w' \in A^+$. We say that the *identity* (or *relation*) $w = w'$ *is satisfied in* S if $\varphi(w) = \varphi(w')$ for every morphism $\varphi : A^+ \to S$.

Notice that we can regard an identity $w = w'$ $(w, w' \in A^+)$ on a semigroup S as a subset of $A^+ \times A^+$.

If $\mathcal{K}$ is a class of semigroups, we say that $\mathcal{K}$ *satisfies an identity* $w = w'$ if each member of $\mathcal{K}$ satisfies $w = w'$. Moreover, if Σ is a set of identities, we say that $\mathcal{K}$ *satisfies* Σ if every member of $\mathcal{K}$ satisfies every identity of Σ.

Birkhoff showed that given a variety V of semigroups, there is a set Σ of identities such that V is precisely the class of semigroups satisfying Σ, that is, V is an *equational class*. We put $V = [\Sigma]$.

Notation 2.1.13. If V is a variety, we denote by $\overline{V}$ the dual variety of V, that is, the variety satisfying the dual identities.

Given a class $\mathcal{K} = \{V_i\}_{i \in I}$ of varieties of semigroups such that $V_i = [R_i]$ $(i \in I)$, then $\bigcap \mathcal{K} = [\bigcup_{i \in I} R_i]$. However, the union of varieties need not be a variety. One does have a well defined join of a class $\mathcal{K} = \{V_i\}_{i \in I}$ of varieties: the intersection of all varieties containing every V_i, denoted by $\bigvee_{i \in I} V_i$. It can be difficult to determine a "good" set of relations for the join of a class of varieties. We can however obtain a set of defining identities in the following way. (See **[16]**.)

Definition 2.1.14. A congruence θ on a semigroup S is *fully invariant* if for every morphism $\varphi : S \to S$

$$(a,b) \in \theta \Longleftrightarrow (\varphi(a), \varphi(b)) \in \theta.$$

The set of fully invariant congruences on a semigroup S is closed under taking arbitrary intersections. Given a set of relations $R \subseteq A^+ \times A^+$, we denote by $\nu(R)$ the smallest fully invariant congruence on A^+ containing R. It is the *fully invariant congruence generated by* R.

If $V = [R]$, sometimes we put $\nu(V)$ for $\nu(R)$.

Proposition 2.1.15. *Let $\mathcal{K} = \{V_i\}_{i \in I}$ be a class of varieties such that $V_i = [R_i]$ $(i \in I)$. Then*

$$\bigvee_{i \in I} V_i = [\bigcap_{i \in I} \nu(R_i)].$$

The next statement yields that in any variety of semigroups there are free objects.

Proposition 2.1.16. *Let $R \subseteq A^+ \times A^+$ be a set of relations and let V be a variety of semigroups such that $V = [R]$. Then $A^+/\nu(R) \in V$ and $A^+/\nu(R)$ is the free object in V generated by A.*

Remark 2.1.17. Let V be a variety of semigroups and let $\natural_V$ be the standard map embedding A in $F_A(V)$. Then, according with 2.1.5, for each semigroup $S \in V$ and each map $\alpha : A \to S$ we have the following commutative diagram

$$\begin{tikzcd} A \arrow[r, "\natural_V"] \arrow[d, "\alpha"'] & F_A(V) \arrow[dl, "\psi"] \\ S & \end{tikzcd} \qquad (2.1.17.1)$$

that is, there is a unique morphism $\psi : F_A(V) \to S$ such that $\psi\natural_V = \alpha$. Moreover, ψ is surjective if and only if $\alpha(A)$ generates S.

The rest of this section amounts to some simplifying notation.

Notation 2.1.18. If S and T are semigroups, ρ is an equivalence relation on T and $f : S \to T$ is a map, let ρ_f be the equivalence relation on S defined by

$$x \; \rho_f \; y \;\text{ if }\; f(x) \; \rho \; f(y).$$

Sometimes we still denote by ρ the relation ρ_f.

Notation 2.1.19. If V is a variety of semigroups, we denote by $\mathcal{L}_V$ $[\mathcal{R}_V, \mathcal{J}_V]$ the Green relation $\mathcal{L}$ $[\mathcal{R}, \mathcal{J}]$ on $F_A(V)$. According to 2.1.18, we can extend this notation without confusion in order to interpret $\mathcal{L}_V$ as a relation on A^+: if $u, v \in A^+$, then by $u \; \mathcal{L}_V \; v$ we shall mean $\natural_V(u) \; \mathcal{L}_V \; \natural_V(v)$.

Remark 2.1.20. Let V and W be varieties of semigroups such that $V \subseteq W$. Then by 2.1.17 there is a unique (surjective) morphism $\natural_{W,V} : F_A(W) \to F_A(V)$ such that the following diagram commutes.

$$\begin{tikzcd} A^+ \arrow[r, "\natural_W"] \arrow[d, "\natural_V"'] & F_A(W) \arrow[dl, "\natural_{W,V}"] \\ F_A(V) & \end{tikzcd}$$

Moreover, the following statements are equivalent

(i) $(\mathcal{L}_V)_{\natural_V} = (\mathcal{L}_W)_{\natural_W}$.

(ii) $(\mathcal{L}_V)_{\natural_{W,V}} = \mathcal{L}_W$.

2. Varieties of bands

Fennemore has shown in [9] that the varieties of bands are each determined by one identity in addition to $x^2 = x$. Because we will be considering only varieties of bands we omit $x^2 = x$ and denote by $\mathbf{V}(P = Q)$ the variety of bands $[x^2 = x, P = Q]$, where P and Q are words on the alphabet $X = \{a, d, x, y, x_1, x_2, \ldots\}$. Also, we put $F_A(P = Q)$ for $F_A(V)$ and $\nu(P = Q)$ for $\nu(P = Q, x^2 = x)$.

Following Fennemore, we define the words R_n, S_n and Q_n for $n \geq 2$, as follows

$$
\begin{aligned}
R_2 &= R_2(x_1, x_2, x_3) = x_3 x_2 x_1, \\
R_3 &= R_3(x_1, x_2, x_3) = x_1 x_2 x_3, \\
Q_2 &= Q_2(x_1, x_2, x_3) = x_2 x_3 x_1, \\
Q_3 &= Q_3(x_1, x_2, x_3) = x_1 x_2 x_3 x_1 x_3, \\
S_2 &= S_2(x_1, x_2, x_3) = x_3 x_1 x_2 x_1, \\
S_3 &= S_3(x_1, x_2, x_3) = x_1 x_2 x_3 x_1 x_3 x_2 x_3, \\
R_n &= R_n(x_1, \ldots, x_n) = R_{n-1} x_n, \text{ for } n = 4, 6 \ldots \\
R_n &= R_n(x_1, \ldots, x_n) = x_n R_{n-1}, \text{ for } n = 5, 7 \ldots \\
Q_n &= Q_n(x_1, \ldots, x_n) = Q_{n-1} x_n R_n, \text{ for } n = 4, 6 \ldots \\
Q_n &= Q_n(x_1, \ldots, x_n) = R_n x_n Q_{n-1}, \text{ for } n = 5, 7 \ldots \\
S_n &= S_n(x_1, \ldots, x_n) = S_{n-1} x_n R_n, \text{ for } n = 4, 6 \ldots \\
S_n &= S_n(x_1, \ldots, x_n) = R_n x_n S_{n-1}, \text{ for } n = 5, 7 \ldots
\end{aligned}
$$

The structure of the lattice **LB** of all varieties of bands has been determined by Birjukov [3], Fennemore [9] and Gerhard [10]. It is as shown in Figure 1. Gerhard and Petrich [14] determined a simpler system of identities for the varieties of bands. They also defined inductively an infinite sequence of invariants, used for solving the word problem for those varieties. We will make use of their system of invariants in the next chapter.

We start with a list of standard definitions on words.

Definition 2.2.1. Let A be an alphabet.

(i) The *content* of $w \in A^+$ is the set of letters occurring in w and is denoted by $c(w)$. By definition, $c(1) = \emptyset$.

(ii) The *dual* of $w \in A^+$ is the word obtained from w by reversing the order of the letters. It is denoted by $\overline{w}$.

(iii) If $w \in A^+$, $\sigma(w)$ $[\varepsilon(w)]$ is the last letter to occur in w, in order from the left [right].

(iv) If $w \in A^+$, $s(w)$ is the longest left cut of w containing all but one of the letters of w. By definition, $s(a^m) = 1$ if $a \in A$.

(v) If $w \in A^+$, $e(w)$ is the longest right cut of w containing all but one of the letters of w. By definition, $e(a^m) = 1$ if $a \in A$.

Remark 2.2.2. Notice that if $w \in A^+$, $\varepsilon(w) = \sigma(\overline{w})$ and $e(w) = \overline{s(\overline{w})}$.

We present now the list of invariants to which we have referred.

Definition 2.2.3.[14] For $n \geq 2$ and $t \in \{h, i, \overline{h}, \overline{i}\}$, let t_n be the maps from A^* into itself defined by

$t_n(1) = 1$, for $n \geq 2$ and, for $w \in A^+$,

$h_2(w) =$ the first letter in w,

$i_2(w) = i_2(s(w))\ \sigma(w)$, the variables of w written in order of first occurrence,

$\overline{t_n}(w) = \overline{t_n(\overline{w})}$, for $n \geq 2$ and $t \in \{h, i\}$,

$t_n(w) = t_n(s(w))\ \sigma(w)\ \overline{t}_{n-1}(w)$, for $n \geq 3$ and $t \in \{h, i\}$.

(2.2.4.) Note that if $a \in A$, then $t_n(a^m) = a^{n-1}$ for all $m \geq 1$, $n \geq 2$ and $t \in \{h, i, \overline{h}, \overline{i}\}$.

The following statements concern some properties of the mappings defined above, which will be useful.

Proposition 2.2.5. **[14]** *Let* $t \in \{h_n : n \geq 3\} \cup \{i_n : n \geq 2\}$. *Then*

(i) $ct = c$

(ii) $st = ts$

(iii) $\sigma t = \sigma$.

Proposition 2.2.6. **[14]** *The following statements hold, for any* $w \in A^+$.

(i) $\overline{i}_2(w) = \varepsilon(w)\ \overline{i}_2(e(w))$.

(ii) $\overline{t}_n(w) = t_{n-1}(w)\ \varepsilon(w)\ \overline{t}_n(e(w))$, *for* $n \geq 3$ *and* $t \in \{h, i\}$.

(iii) $t_n(w) = t_n(s(w))\ \sigma(w)\ \varepsilon(w)\ \overline{t}_{n-1}(e(w))$, *for* $n \geq 3$ *if* $t = i$ *and for* $n \geq 4$ *if* $t = h$.

(iv) $\overline{t}_n(w) = t_{n-1}(s(w))\ \sigma(w)\ \varepsilon(w)\ \overline{t}_n(e(w))$, *for* $n \geq 3$ *if* $t = i$ *and for* $n \geq 4$ *if* $t = h$.

Proposition 2.2.7. **[14]** *Let* $n \geq 2$ *and* $u, v \in A^+$.

(i) *If* $h_{n+1}(u) = h_{n+1}(v)$, *then* $\overline{h}_n(u) = \overline{h}_n(v)$ *and* $i_n(u) = i_n(v)$.

(ii) *If* $i_n(u) = i_n(v)$, *then* $h_n(u) = h_n(v)$.

(iii) *If* $i_{n+1}(u) = i_{n+1}(v)$, *then* $\overline{i}_n(u) = \overline{i}_n(v)$.

From Proposition 2.2.7 we deduce the following statement.

Proposition 2.2.8. *Let $n \geq 2$ and $u, v \in A^+$. If $t \in \{h, i\}$, then*

$$t_{n+1}(u) = t_{n+1}(v) \Longrightarrow t_n(u) = t_n(v).$$

Proposition 2.2.9. *Let $n \geq 3$, $u, v \in A^+$ and $t \in \{h, i\}$.*

(i) *$t_n(u) = t_n(v)$ if and only if $t_n(s(u)) = t_n(s(v))$, $\sigma(u) = \sigma(v)$ and $\overline{t}_{n-1}(u) = \overline{t}_{n-1}(v)$.*

(ii) *$\overline{t}_n(u) = \overline{t}_n(v)$ if and only if $\overline{t}_n(e(u)) = \overline{t}_n(e(v))$, $\varepsilon(u) = \varepsilon(v)$ and $t_{n-1}(u) = t_{n-1}(v)$.*

(iii) *$t_n(u) = t_n(v)$ if and only if $t_n(s(u)) = t_n(s(v))$, $\sigma(u) = \sigma(v)$, $\varepsilon(u) = \varepsilon(v)$ and $\overline{t}_{n-1}(e(u)) = \overline{t}_{n-1}(e(v))$, for $n \geq 3$ if $t = i$ and for $n \geq 4$ if $t = h$.*

(iv) *$\overline{t}_n(u) = \overline{t}_n(v)$ if and only if $t_{n-1}(s(u)) = t_{n-1}(s(v))$, $\sigma(u) = \sigma(v)$, $\varepsilon(u) = \varepsilon(v)$ and $\overline{t}_n(e(u)) = \overline{t}_n(e(v))$, for $n \geq 3$ if $t = i$ and for $n \geq 4$ if $t = h$.*

Proof. (i) The direct part for $t = h_3$ is trivial. If $t \neq h_3$, then from the definition and 2.2.5 (ii) and (iii), we get that

$$t_n(u) = t_n(v) \Rightarrow t_n(s(u)) = t_n(s(v)) \quad \text{and} \quad \sigma(u) = \sigma(v).$$

Also, from 2.2.7 (i) and (iii) it follows that

$$t_n(u) = t_n(v) \Rightarrow \overline{t}_{n-1}(u) = \overline{t}_{n-1}(v).$$

The converse follows by definition.

(ii) The proof is dual to the proof of (i).

(iii) and (iv) follow immediately from (i), (ii) and 2.2.6(ii), (iv).

The following result is essentially Proposition 9.1 of [**14**].

Proposition 2.2.10. *Let* $u, v \in A^+$.

(i) $(u,v) \in \nu(xy = yx) \iff c(u) = c(v)$.

(ii) $(u,v) \in \nu(ax = a) \iff h_2(u) = h_2(v)$.

(iii) $(u,v) \in \nu(ax = axa) \iff i_2(u) = i_2(v)$.

(iv) *For* $n \geq 3$

$(u,v) \in \nu(R_n = Q_n) \iff h_n(u) = h_n(v)$, *for n odd.*

$(u,v) \in \nu(R_n = Q_n) \iff \overline{h}_n(u) = \overline{h}_n(v)$, *for n even.*

$(u,v) \in \nu(R_n = S_n) \iff i_n(u) = i_n(v)$, *for n odd.*

$(u,v) \in \nu(R_n = S_n) \iff \overline{i}_n(u) = \overline{i}_n(v)$, *for n even.*

Remark 2.2.11. Notice that by 2.1.15 the solution of the word problem for the join of two varieties is just the conjunction of the solutions of the word problem for the two varieties. For instance, we have

$$\begin{aligned}(u,v) \in \nu(\overline{R}_2 = \overline{Q}_2) &\iff (u,v) \in \nu(ax = a) \cap \nu(xy = yx)\\ &\iff h_2(u) = h_2(v), c(u) = c(v).\end{aligned}$$

Remark 2.2.12. Let $\mathcal{P}_{\text{fin}}(A)$ be the set of finite subsets of A. The content can be regarded as a map $c : A^+ \to \mathcal{P}_{\text{fin}}(A) \setminus \emptyset$. Therefore, by 2.2.10 $(\mathcal{P}_{\text{fin}}(A) \setminus \emptyset, \cup)$ is the free semilattice generated by A.

For example if $A = \{a, b\}$ then $F_A(xy = yx) = \{\{a\}, \{b\}, \{a, b\}\}$.

In general we identify $\{a\}$ with a, $\{b\}$ with b, $\{a, b\}$ with ab and write simply $F_A(xy = yx) = \{a, b, ab\}$.

Remark 2.2.13. Let $X = \{(B, b) : b \in B \subseteq A, B \text{ finite}\}$ and define a multiplication $\circ$ in X by

$$(B, b) \circ (C, c) = (B \cup C, b).$$

Then $(X, \circ)$ is a semigroup. Define a mapping $f : A^+ \to X$ by $f(u) = (c(u), h_2(u))$. We have a well defined mapping $\hat{f} : F_A(\overline{R}_2 = \overline{Q}_2) \to X$ defined by

$$\hat{f}\big([u]_{\nu(\overline{R}_2 = \overline{Q}_2)}\big) = f(u).$$

Clearly $\hat{f}$ is an isomorphism and 2.2.10 yields that $(X, \circ)$ is the free object in $\mathbf{V}(\overline{R}_2 = \overline{Q}_2)$ generated by A.

Remark 2.2.14. [14] (i) The semigroup $i_2(A^+)$ with multiplication

$$i_2(u) \circ i_2(v) = i_2(uv)$$

is the free object in $\mathbf{V}(ax = axa)$ generated by A.

(ii) For $n \geq 3$, n odd, the semigroup $h_n(A^+)$ with multiplication

$$h_n(u) \circ h_n(v) = h_n(uv)$$

is the free object in $\mathbf{V}(R_n = Q_n)$ generated by A.

(iii) For $n \geq 3$, n odd, the semigroup $i_n(A^+)$ with multiplication

$$i_n(u) \circ i_n(v) = i_n(uv)$$

is the free object in $\mathbf{V}(R_n = S_n)$ generated by A.

Remark 2.2.15. From 1.5.5 we have that the varieties $\mathbf{V}(x = y), \mathbf{V}(ax = a), \mathbf{V}(xa = a)$ and $\mathbf{V}(axa = a)$ are subvarieties of the variety $\mathbf{V}(axa = a)$ of rectangular bands. We call $\mathbf{LB_0}$ the lattice obtained from $\mathbf{LB}$ by taking away these varieties. From 2.2.5, 2.2.10 and 2.2.11 we deduce that for every variety $V \in \mathbf{LB_0}$

$$(u, v) \in \nu(V) \Rightarrow c(u) = c(v) \quad (u, v \in A^+).$$

Thus, it is possible to talk in an unambiguous way about the content of an element of $F_A(V)$ ($V \in \mathbf{LB_0}$).

The previous remark will determine the strategy used in the remainder of the thesis: we treat first the case of the varieties of rectangular bands and then we work on $\mathbf{LB_0}$.

The following result will be useful, too. (Recall 2.1.19 for the notation used.)

Proposition 2.2.16. **[16]** *Given $V \in \mathbf{LB_0}$ and $u, v \in A^+$, we have*

$$(u, v) \in \mathcal{J}_V \Longleftrightarrow c(u) = c(v).$$

This means that in the free objects of the varieties of $\mathbf{LB_0}$ the $\mathcal{J}$-classes are defined by the content.

Corollary 2.2.17. *If $V, W \in \mathbf{LB_0}$, then $\mathcal{J}_V = \mathcal{J}_W$.* (See 2.1.19 for the notation.)

We will introduce the following distinction between the varieties of $\mathbf{LB_0}$.

Definition 2.2.18. The varieties of $\mathbf{LB_0}$, $\mathbf{V}(ax = axa)$, $\mathbf{V}(R_n = T_n)$, $\mathbf{V}(\overline{R}_m = \overline{T}_m)$, $T \in \{Q, S\}$, n odd, m even, $n, m \geq 3$, are called *strictly left varieties*. The duals of these varieties are called *strictly right varieties*.

The varieties of $\mathbf{LB_0}$, $\mathbf{V}(xy = yx)$, $\mathbf{V}(\overline{R}_2 = \overline{Q}_2)$ and $\mathbf{B}$, together with the strictly left varieties, are called *left varieties*. The duals of these varieties are called *right varieties*. (See Figure 3).

The following notation will be useful.

Notation 2.2.19. (1) If $V \in \mathbf{LB_0}$ is a proper variety, we denote by V_l $[V_r]$ the maximum of the set of left [right] varieties contained in V. Then

$$V = V_l \vee V_r.$$

(2) If $V \in \mathbf{LB_0}$ is a proper variety, we denote by V^l $[V^r]$ the minimum of the set of strictly left [strictly right] varieties containing V.

We define $\mathbf{B}^r = \mathbf{B}^l = \mathbf{B}$. Then if $V \in \mathbf{LB_0}$ we have

$$V = V^l \cap V^r.$$

For example, if $V = \mathbf{V}(\overline{R}_2 = \overline{S}_2)$, then $V_l = \mathbf{V}(ax = axa)$, $V_r = \mathbf{V}(R_2 = Q_2)$, $V^l = \mathbf{V}(R_3 = Q_3)$ and $V^r = \mathbf{V}(\overline{R}_3 = \overline{S}_3)$. If $V = \mathbf{V}(xy = yx)$, then $V_l = V_r = V$, $V^l = \mathbf{V}(ax = axa)$ and $V^r = \mathbf{V}(xa = axa)$.

3. Varieties of band monoids

Wismath [28] determined the structure of the lattice **LBM** of varieties of band monoids. This lattice is as shown in Figure 2.

Let Mon : $\mathbf{LB} \longrightarrow \mathbf{LBM}$ be the map defined by

$$\mathrm{Mon}(V) = V \cap \mathbf{M}$$

Proposition 2.3.1. [Wismath, 28] Mon *is a lattice morphism.*

Let $\chi : \mathbf{LBM} \longrightarrow \mathbf{LB}$ be the map defined by

$$\chi(V) = < V >_S$$

where $< V >_{\mathbf{S}}$ is the variety of bands generated by V. (See 2.1.2.)

We have

Proposition 2.3.2. *χ is a lattice morphism.*

Proof. Let $V_1, V_2 \in \mathbf{LBM}$. Then

$$\begin{aligned}\chi(V_1 \vee V_2) &=< V_1 \vee V_2 >_{\mathbf{S}} \\ &=<< V_1 \cup V_2 >_{\mathbf{M}} >_{\mathbf{S}} \\ &=< V_1 \cup V_2 >_{\mathbf{S}} \qquad \text{by 2.1.4} \\ &=< V_1 >_{\mathbf{S}} \vee < V_2 >_{\mathbf{S}} \\ &= \chi(V_1) \vee \chi(V_2).\end{aligned}$$

Now we notice that if $V \in \mathbf{LBM}$, then

$$\chi(V) = \cap\{W \in \mathbf{LB} : \mathrm{Mon}(W) \supseteq V\}$$

and then

$$\begin{aligned}\chi(V_1 \cap V_2) &= \cap\{W \in \mathbf{LB} : \mathrm{Mon}(W) \supseteq (V_1 \cap V_2)\} \\ &= \big(\cap\{W \in \mathbf{LB} : \mathrm{Mon}(W) \supseteq V_1\}\big) \cap \big(\cap\{W \in \mathbf{LB} : \mathrm{Mon}(W) \supseteq V_2\}\big) \\ &= \chi(V_1) \cap \chi(V_2).\end{aligned}$$

The following result is essentially contained in Wismath.

Proposition 2.3.3. $\mathrm{Mon} \circ \chi = \mathrm{id}_{\mathbf{LBM}}$.

Proof. Let V be a variety of band monoids. Then

$$\mathrm{Mon} \circ \chi(V) = < V >_{\mathbf{S}} \cap \mathbf{M} \supseteq V.$$

Conversely, let $T \in < V >_{\mathbf{S}} \cap \mathbf{M}$. By 2.1.3, T divides a direct product $\prod_{i\in I} S_i$, where the S_i $(i \in I)$ are monoids of V.

Let $M = \prod_{i\in I} S_i$. There is a subsemigroup N of M and there is a surjective morphism $\varphi : N \to T$. If N is a submonoid of M then $N \in V$, φ is a monoid morphism and hence $T \in V$.

If N is not a submonoid of M then $N^{\mathrm{I}} = N \cup \{\mathrm{id}_M\}$ is a submonoid of M and $\varphi^{\mathrm{I}} : N^{\mathrm{I}} \to T$ defined by

$$\varphi^{\mathrm{I}}(x) = \begin{cases} \varphi(x), & \text{if } x \in N; \\ \mathrm{id}_T, & \text{if } x = \mathrm{id}_N. \end{cases}$$

is a surjective monoid morphism and we deduce again that $T \in V$.

(2.3.4.) We denote by $\mathbf{VM}(P = Q)$ the variety of band monoids satisfying the identity $P = Q$. We note that $\mathrm{Mon}(\mathbf{V}(P = Q)) = \mathbf{VM}(P = Q)$. Also, Proposition 2.3.3 yields in particular that $\chi(\mathbf{LBM}) \simeq \mathbf{LBM}$. For example, we have

$$\mathrm{Mon}(\mathbf{V}(R_3 = S_3)) = \mathrm{Mon}(\mathbf{V}(\overline{R}_4 = \overline{Q}_4)) = \mathbf{VM}(R_3 = S_3),$$

since

$$\mathbf{VM}(R_3 = S_3) = \mathbf{VM}(\overline{R}_4 = \overline{Q}_4). \quad \text{(See [28].)}$$

Also,

$$\chi(\mathbf{VM}(R_3 = S_3)) = \mathbf{V}(R_3 = S_3).$$

Let $\mathbf{LBM_0}$ be the lattice obtained from $\mathbf{LBM}$ by taking away the trivial variety $\mathbf{VM}(x = 1)$.

As we have done for $\mathbf{LB_0}$ (see 2.2.18), we introduce the following distinction between the left and right "parts" of $\mathbf{LBM_0}$.

Definition 2.3.5. The varieties of $\mathbf{LBM_0}$, $\mathbf{VM}(ax = axa)$, $\mathbf{VM}(R_n = S_n)$, n odd, $n \geq 3$, $\mathbf{VM}(\overline{R}_m = \overline{S}_m)$, m even, $m \geq 4$ are called *strictly left varieties*. The duals of these varieties are called *strictly right varieties*. The varieties $\mathbf{VM}(xy = yx)$, **BM**, together with the strictly left varieties are called *left varieties*. The duals of these varieties are called *right varieties.*

As for bands, if $V \in \mathbf{LBM_0}$ is a proper variety, we denote by V_l $[V_r]$ the maximum of the set of left [right] varieties of band monoids contained in V and we denote by V^l $[V^r]$ the minimum of the set of strictly left [strictly right] varieties of band monoids containing V. We define $\mathbf{BM}^l = \mathbf{BM}^r = \mathbf{BM}$.

4. Eilenberg varieties

Since the direct product of a family of finite semigroups need not be a finite semigroup, we cannot use the notion of Birkhoff variety to study the class of finite semigroups. However, the finite case is important, namely for its link with the theory of formal languages and the theory of finite automata. Eilenberg introduced a different notion of variety of semigroups, dealing this way with the finite case.

Definition 2.4.1. A *variety of finite semigroups* [*monoids*] or *Eilenberg variety of semigroups* [*monoids*] is a class of finite semigroups [monoids] closed under taking subsemigroups [submonoids], quotients and finite direct products.

Thus, Eilenberg's definition differs from the general definition of variety of universal algebras, due to Birkhoff, by authorising only finite direct products.

The intersection of any family of Eilenberg varieties of semigroups [monoids] is again an Eilenberg variety of semigroups [monoids]. Hence, if C is a class of finite semigroups [monoids] we call the *variety generated by* C, denoted by (C), the intersection of all Eilenberg varieties of semigroups [monoids] containing C. If $C = \{M\}$, we shall use the notation (M) instead of $(\{M\})$.

The Eilenberg varieties of semigroups are not necessarily equational classes, as happens with Birkhoff varieties. We have a weaker result.

Definition 2.4.2. Let $(u_n, v_n)_{n>0}$ be a sequence of pairs of words of A^+ and let V be an Eilenberg variety of semigroups [monoids]. We say that V is *ultimately defined* by the identities $u_n = v_n$ $(n > 0)$ if for all semigroup [monoid] S, S is in V if and only if it satisfies the identity $u_n = v_n$ for every n sufficiently large.

Eilenberg and Schützenberger (1975) proved the following result.

Theorem 2.4.3. *Every Eilenberg variety of semigroups* [*monoids*] *is ultimately defined by a sequence of equations.*

However, there are some Eilenberg varieties of semigroups [monoids] which are defined by a sequence of equations. This is the case of varieties which are generated by a single semigroup [monoid], as happens with the varieties of the finite band monoids (as we shall see).

Indeed, the finite band monoids form a variety in the sense of Eilenberg. It was Wismath [28] who determined the structure of the lattice **LFBM** of finite band monoids. She defined a map Fin from the lattice **LBM** of band monoids (pictured in

Figure 2) to the lattice **LFBM** by

$$\mathrm{Fin}(V) = \text{ the set of finite monoids in } V$$

and showed that Fin is a lattice isomorphism from **LBM** to **LFBM**. Therefore, the lattice of Eilenberg varieties of band monoids is still the one pictured in Figure 2.

Remark 2.4.4. The free band on a finite set of generators is finite. Consequently, the finitely generated free objects of the varieties of bands (in the sense of Birkhoff) are also finite. The same holds for band monoids. Therefore, the finitely generated free objects in the varieties of band monoids are free objects in the corresponding Eilenberg varieties of band monoids, by the map Fin. We shall use this fact in the third part of this thesis.

For an extensive discussion of Eilenberg varieties of semigroups, we refer to Eilenberg **[8]**, Lallement **[18]** or Pin **[20]**.

CHAPTER 3. THE CARDINALITIES OF THE GREEN CLASSES OF THE FREE OBJECTS IN VARIETIES OF BANDS

In this chapter we work with Birkhoff varieties of bands.

The cardinalities of the Green classes of the free objects in each variety of **LB** constitute a useful piece of information in the study of several questions about these varieties. We compute these cardinalities in section 1. In section 2 we analyse the following question: given two varieties V and W such that $V \subseteq W$, for which cardinalities of A do we have $F_A(V) = F_A(W)$?

The last section consists of a brief analysis which enables us to deduce, for varieties of band monoids, results corresponding to the ones obtained for varieties of bands.

1. The cardinalities of the Green classes of the free objects in varieties of bands

We aim to be able to compute the cardinalities of the Green classes of the free objects $F_A(V)$, for $V \in \mathbf{LB}$.

We study first the varieties of rectangular bands. By 1.5.8 if V is a variety of rectangular bands then $F_A(V)$ has just one $\mathcal{J}$-class, with cardinality $|\ F_A(V)\ |$. Let us see what we get in each case.

Let $V = \mathbf{V}(ax = a)$ and let $u, v \in A^+$. According with the notation introduced in 2.1.19, we have

$$\begin{aligned} u\mathcal{L}_V v &\Longleftrightarrow (uv,u),(vu,v) \in \nu(ax=a), \quad \text{by 1.5.1} \\ &\Longleftrightarrow \begin{cases} h_2(uv) = h_2(u) \\ h_2(vu) = h_2(v) \end{cases} \quad \text{by 2.2.10} \\ &\Longleftrightarrow \begin{cases} h_2(u) = h_2(u) \\ h_2(v) = h_2(v) \end{cases} \end{aligned}$$

Thus, all elements are $\mathcal{L}$-related; that is, there is only one $\mathcal{L}$-class, which is also the only $\mathcal{J}$-class, with cardinality $|A|$.

We consider now the case of $V = \mathbf{V}(axa = a)$. Since

$$\mathbf{V}(axa = a) = \mathbf{V}(ax = a) \vee \mathbf{V}(xa = a)$$

we get

$$(u,v) \in \nu(V) \Longleftrightarrow \begin{cases} h_2(u) = h_2(v) \\ \overline{h}_2(u) = \overline{h}_2(v) \end{cases}$$

that is, u and v are equivalent if and only if both their first letters and their last letters coincide. Therefore $F_A(V)$ is equipotent to $A \times A$ and since $F_A(V)$ is isomorphic with its dual, each $\mathcal{L}$-class and each $\mathcal{R}$-class have the same cardinality, which is $|A|$.

From now on we work on $\mathbf{LB_0}$.

For each non-empty finite subset X of A and for $V \in \mathbf{LB_0}$, we denote by $U_X(V)$ the set of elements in $F_A(V)$ with content X. By 2.2.16, each $U_X(V)$ is a $\mathcal{J}$- class of $F_A(V)$. If $|X| = k$, we denote by $c_k(V)$ the cardinality of $U_X(V)$.

Clearly, $c_1(V) = 1$ and $c_k(V) = c_k(\overline{V})$. (See 2.1.13 for the notation.)

If $V = \mathbf{V}(P = Q)$, sometimes we put $c_k(P = Q)$ for $c_k(V)$.

Remark 3.1.1. If A is finite — say $|A| = N$ — and $V \in \mathbf{LB_0}$, the free objects $F_A(V)$ are finite and

$$|F_A(V)| = \sum_{k=1}^{N} \binom{N}{k} c_k(V).$$

Example 3.1.2. Let $V = \mathbf{V}(xy = yx)$. By 2.2.10, it is clear that $c_k(xy = yx) = 1$, for all $k \geq 1$. Then if $A = \{a, b, c\}$

$$| F_A(V) | = \binom{3}{1} c_1(V) + \binom{3}{2} c_2(V) + \binom{3}{3} c_3(V) = 7$$

Indeed, $F_A(V) = \{a, b, c, ab, ac, bc, abc\}$. (See 2.2.12.)

Remark 3.1.3. Let $V, W \in \mathbf{LB_0}$ and let $X \subseteq A$. If $V \subseteq W$, then $U_X(V)$ is an epimorphic image of $U_X(W)$, by the canonical epimorphism $\natural_{W,V} : F_A(W) \rightarrow F_A(V)$. Hence, if $|X| = k$,

$$c_k(V) \leq c_k(W).$$

Remark 3.1.4. Let $V \in \mathbf{LB_0}$. As we have pointed out in 1.5.9, the cardinality of an $\mathcal{R}$ $[\mathcal{L}]$-class in a certain $\mathcal{J}$-class of $F_A(V)$ is just the number of $\mathcal{L}$ $[\mathcal{R}]$-classes contained in the $\mathcal{J}$-class. If $X \subseteq A$, $|X| = k$, we denote by $R_k(V)$ $[L_k(V)]$, the number of $\mathcal{L}$ $[\mathcal{R}]$-classes in the $\mathcal{J}$-class $U_X(V)$. Clearly $L_1(V) = R_1(V) = 1$.

If $V = \mathbf{V}(P = Q)$, sometimes we put $R_k(P = Q)$ $[L_k(P = Q)]$ for $R_k(V)$ $[L_k(V)]$.

(3.1.5.) Green and Rees proved that each $\mathcal{J}$-class of a free band $F_A(\mathbf{B})$ has cardinality $c_k(\mathbf{B})$, where $c_k(\mathbf{B})$ is defined inductively by

$$c_k(\mathbf{B}) = k^2 c_{k-1}^2(\mathbf{B}). \quad \text{(See [16, chapter 4].)}$$

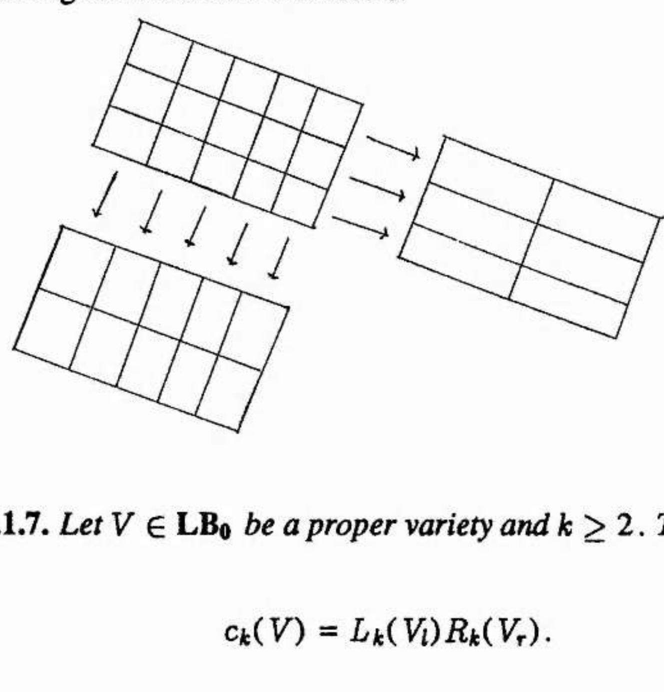
Since $F_A(\mathbf{B})$ is isomorphic to its dual, if $J = U_X$, $|X| = k$, is a $\mathcal{J}$-class of $F_A(\mathbf{B})$, we deduce that the number of $\mathcal{R}$-classes of J is equal to the number of $\mathcal{L}$-classes of J. Therefore, the cardinality of an $\mathcal{L}$-class of J is equal to the cardinality of an $\mathcal{R}$-class of J and is $\sqrt{c_k(\mathbf{B})} = kc_{k-1}(\mathbf{B})$.

We present now the main result of this section and a corollary. (For the notations V_l, V_r, see 2.2.19).

Theorem 3.1.6. *Let $V \in \mathbf{LB_0}$ be a proper variety. Then the number of $\mathcal{L}$ $[\mathcal{R}]$ - classes of a $\mathcal{J}$-class $U_X(V)$ is equal to the number of $\mathcal{L}$ $[\mathcal{R}]$ - classes of $U_X(V_r)$ $[U_X(V_l)]$; that is,*

$$R_k(V) = R_k(V_r) \quad ; \quad L_k(V) = L_k(V_l)$$

The next diagram illustrates this result.



Corollary 3.1.7. *Let $V \in \mathbf{LB_0}$ be a proper variety and $k \geq 2$. Then*

$$c_k(V) = L_k(V_l)R_k(V_r).$$

Theorem 3.1.6 follows immediately from the next two results, concerning the study of the relations $\mathcal{R}$ and $\mathcal{L}$ in the free objects of the varieties of $\mathbf{LB_0}$. In the first one, we analyse some particular cases.

Lemma 3.1.8. (i) *If V is one of the varieties $\mathbf{V}(\overline{R}_2 = \overline{Q}_2), \mathbf{V}(ax = axa)$, then*

$$\mathcal{R}_V = \mathrm{id}_{F_A(V)}$$

and

$$\mathcal{L}_V = \mathcal{J}_V = \mathcal{L}_{\mathbf{V}(xy=yx)}.$$

(ii) *If $V = \mathbf{V}(axya = ayxa)$, then*

$$\mathcal{R}_V = \mathcal{R}_{\mathbf{V}(\overline{R}_2=\overline{Q}_2)} = \nu(\overline{R}_2 = \overline{Q}_2)$$

and

$$\mathcal{L}_V = \mathcal{L}_{\mathbf{V}(R_2=Q_2)} = \nu(R_2 = Q_2).$$

(iii) $V = \mathbf{V}(\overline{R}_2 = \overline{S}_2)$, *then*

$$\mathcal{R}_V = \mathcal{R}_{\mathbf{V}(ax=axa)} = \nu(ax = axa)$$

and

$$\mathcal{L}_V = \mathcal{L}_{\mathbf{V}(R_2=Q_2)} = \nu(R_2 = Q_2).$$

Proof. (i) Let $u, v \in A^+$.

If $V = \mathbf{V}(\overline{R}_2 = \overline{Q}_2)$, then

$$\begin{aligned}
(u,v) \in \mathcal{R}_V &\iff (uv,v),(vu,u) \in \nu(\overline{R}_2 = \overline{Q}_2) \qquad \text{by 1.5.1(ii)} \\
&\iff (uv,v),(vu,u) \in \nu(ax = a) \cap \nu(xy = yx) \\
&\iff \begin{cases} h_2(uv) = h_2(v) \\ c(uv) = c(v) \\ h_2(vu) = h_2(u) \\ c(vu) = c(u) \end{cases} \quad \text{by 2.2.11(i),(ii)} \\
&\iff \begin{cases} h_2(u) = h_2(v) \\ c(u) = c(v) \end{cases} \\
&\iff (u,v) \in \nu(\overline{R}_2 = \overline{Q}_2).
\end{aligned}$$

Thus,

$$\mathcal{R}_V = \mathrm{id}_{F_A(V)}$$

and

$$\begin{aligned}\mathcal{L}_V &= \mathcal{J}_V \quad \text{by 1.5.2}\\ &= \mathcal{L}_{\mathbf{V}(xy=yx)} \quad \text{by 2.2.16 and 2.2.10(i).}\end{aligned}$$

If $V = \mathbf{V}(ax = axa)$, then

$$\begin{aligned}(u,v) \in \mathcal{R}_V &\Longleftrightarrow (uv,v),(vu,u) \in \nu(ax = axa) \quad \text{by 1.5.1(ii)}\\ &\Longleftrightarrow \begin{cases} i_2(uv) = i_2(v)\\ i_2(vu) = i_2(u)\end{cases} \quad \text{by 2.2.10(iii)}\\ &\Longleftrightarrow i_2(u) = i_2(v)\\ &\Longleftrightarrow (u,v) \in \nu(ax = axa).\end{aligned}$$

Hence,

$$\mathcal{R}_V = \mathrm{id}_{F_A(V)}$$

and

$$\mathcal{L}_V = \mathcal{J}_V = \mathcal{L}_{\mathbf{V}(xy=yx)}.$$

(ii) For $u, v \in A^+$ and $V = \mathbf{V}(axya = ayxa)$, we have

$$\begin{aligned}(u,v) \in \mathcal{R}_V &\Longleftrightarrow (uv,v),(vu,u) \in \nu(axya = ayxa) \quad \text{by 1.5.1(ii)}\\ &\Longleftrightarrow (uv,v),(vu,u) \in \nu(R_2 = Q_2) \cap \nu(\overline{R}_2 = \overline{Q}_2)\\ &\Longleftrightarrow \begin{cases} h_2(uv) = h_2(v)\\ c(uv) = c(v)\\ h_2(vu) = h_2(u)\\ c(vu) = c(u)\\ \overline{h}_2(uv) = \overline{h}_2(v)\\ \overline{h}_2(vu) = \overline{h}_2(u)\end{cases}\\ &\Longleftrightarrow \begin{cases} h_2(u) = h_2(v)\\ c(u) = c(v)\end{cases}\\ &\Longleftrightarrow (u,v) \in \nu(\overline{R}_2 = \overline{Q}_2).\end{aligned}$$

Similarly, we deduce that

$$\mathcal{L}_V = \mathcal{L}_{\mathbf{V}(R_2=Q_2)} = \nu(R_2 = Q_2).$$

(iii) For $u, v \in A^+$ and $V = \mathbf{V}(\overline{R}_2 = \overline{S}_2)$, we have

$$\begin{aligned}
(u,v) \in \mathcal{R}_V &\Longleftrightarrow (uv, v), (vu, u) \in \nu(\overline{R}_2 = \overline{S}_2) \quad \text{by 1.5.1(ii)} \\
&\Longleftrightarrow (uv, v), (vu, u) \in \nu(ax = axa) \cap \nu(R_2 = Q_2) \\
&\Longleftrightarrow \begin{cases} i_2(uv) = i_2(v) \\ i_2(vu) = i_2(u) \\ \overline{h}_2(uv) = \overline{h}_2(v) \\ \overline{h}_2(vu) = \overline{h}_2(u) \end{cases} \\
&\Longleftrightarrow i_2(u) = i_2(v) \\
&\Longleftrightarrow (u,v) \in \nu(ax = axa) \\
&\Longleftrightarrow (u,v) \in \mathcal{R}_{\mathbf{V}(ax=axa)}.
\end{aligned}$$

Now,

$$\begin{aligned}
(u,v) \in \mathcal{L}_V &\Longleftrightarrow (uv, u), (vu, v) \in \nu(ax = axa) \cap \nu(R_2 = Q_2) \quad \text{by 1.5.1(i)} \\
&\Longleftrightarrow \begin{cases} i_2(uv) = i_2(u) \\ i_2(vu) = i_2(v) \\ \overline{h}_2(uv) = \overline{h}_2(u) \\ \overline{h}_2(vu) = \overline{h}_2(v) \end{cases} \\
&\Longleftrightarrow \begin{cases} c(u) = c(v) \\ \overline{h}_2(u) = \overline{h}_2(v) \end{cases} \\
&\Longleftrightarrow (u,v) \in \nu(R_2 = Q_2) \\
&\Longleftrightarrow (u,v) \in \mathcal{L}_{\mathbf{V}(R_2=Q_2)}.
\end{aligned}$$

In general, we have

Proposition 3.1.9. *Let $V = V_l \vee V_r$. Then*

$$\mathcal{R}_V = \mathcal{R}_{V_l} \quad and \quad \mathcal{L}_V = \mathcal{L}_{V_r}.$$

Proof. If $V = \mathbf{V}(xy = yx)$, then $V_l = V_r = V$ and the statement is trivially true. Otherwise, let $u, v \in A^+$. Then

$$(u,v) \in \nu(V_l) \Longleftrightarrow \begin{cases} c(u) = c(v) \\ t_n(u) = t_n(v) \end{cases}$$

and

$$(u,v) \in \nu(V_r) \Longleftrightarrow \begin{cases} c(u) = c(v) \\ \overline{q}_m(u) = \overline{q}_m(v), \end{cases}$$

for some $t, q \in \{h, i\}$ and some $n, m \geq 2$, $|n - m| \leq 1$. We list the cases that remain to be considered.

(1) $t = q = i, n = m = 2$.

(2) $m = n, n \geq 3$ and $t, q \in \{h, i\}$.

(3) $t = q, t, q \in \{h, i\}, m = n - 1, n \geq 3$.

(4) $m = n - 1, n \geq 3, t = h, q = i$.

We now prove (1) – (4).

(1) This is the case of $V_l = \mathbf{V}(ax = axa), V_r = \mathbf{V}(xa = axa)$ and $V = \mathbf{V}(axya = axaya)$.

We get

$$\begin{aligned} (u,v) \in \mathcal{R}_V &\Longleftrightarrow (uv, v), (vu, u) \in \nu(ax = axa) \cap \nu(xa = axa) \\ &\Longleftrightarrow \begin{cases} i_2(uv) = i_2(v) \\ i_2(vu) = i_2(u) \\ \overline{i}_2(uv) = \overline{i}_2(v) \\ \overline{i}_2(vu) = \overline{i}_2(u) \end{cases} \end{aligned}$$

$$\Longleftrightarrow i_2(u) = i_2(v)$$

$$\Longleftrightarrow (u,v) \in \nu(ax = axa)$$

$$\Longleftrightarrow (u,v) \in \mathcal{R}_{\mathbf{V}(ax=axa)}.$$

Dually, we get

$$\begin{aligned}(u,v) \in \mathcal{L}_V &\Longleftrightarrow \bar{i}_2(u) = \bar{i}_2(v)\\ &\Longleftrightarrow (u,v) \in \nu(xa = axa)\\ &\Longleftrightarrow (u,v) \in \mathcal{L}_{\mathbf{V}(xa=axa)}.\end{aligned}$$

(2) For n odd, this is the case of $V_l = \mathbf{V}(R_n = T_n)$, $V_r = \mathbf{V}(\overline{R}_n = \overline{T}_n)$ and for n even, this is the case of $V_l = \mathbf{V}(\overline{R}_n = \overline{P}_n)$, $V_r = \mathbf{V}(R_n = P_n)$, where $T, P \in \{Q, S\}$. In both cases, we get

$$(u,v) \in \mathcal{R}_V \Longleftrightarrow \begin{cases} t_n(uv) = t_n(v) \\ \overline{q}_n(uv) = \overline{q}_n(v) \\ t_n(vu) = t_n(u) \\ \overline{q}_n(vu) = \overline{q}_n(u) \end{cases}$$

$$\Longleftrightarrow \begin{cases} t_n(uv) = t_n(v) \\ \overline{q}_n e(uv) = \overline{q}_n e(v) \\ \varepsilon(uv) = \varepsilon(v) \\ q_{n-1}(uv) = q_{n-1}(v) \\ t_n(vu) = t_n(u) \\ \overline{q}_n e(vu) = \overline{q}_n e(u) \\ \varepsilon(vu) = \varepsilon(u) \\ q_{n-1}(vu) = q_{n-1}(u) \end{cases} \quad \text{by 2.2.9(iv)}$$

$$\Longleftrightarrow \begin{cases} t_n(uv) = t_n(v) \\ q_{n-1}(uv) = q_{n-1}(v) \\ t_n(vu) = t_n(u) \\ q_{n-1}(vu) = q_{n-1}(u) \end{cases}$$

$$\Longleftrightarrow \begin{cases} t_n(uv) = t_n(v) \\ t_n(vu) = t_n(u) \end{cases} \quad \text{by 2.2.7 and 2.2.8}$$

$$\Longleftrightarrow (u,v) \in \mathcal{R}_{V_l}.$$

Dually, we get

$$(u,v) \in \mathcal{L}_V \iff \begin{cases} \overline{q}_n(uv) = \overline{q}_n(u) \\ \overline{q}_n(vu) = \overline{q}_n(v) \end{cases}$$
$$\iff (u,v) \in \mathcal{L}_{V_r}.$$

(3) For $t = i$ and $n = 3$, $V_l = \mathbf{V}(R_3 = S_3)$ and $V_r = \mathbf{V}(ax = axa)$.

For n odd, $n > 3$, this is the case of $V_l = \mathbf{V}(R_n = T_n)$ and $V_r = \mathbf{V}(R_{n-1} = T_{n-1})$ and for n even, this is the case of $V_l = \mathbf{V}(\overline{R}_n = \overline{T}_n)$ and $V_r = \mathbf{V}(\overline{R}_{n-1} = \overline{T}_{n-1})$, where $T \in \{Q, S\}$.

We notice first that in all cases $V_l \vee V_r = V_l$. Also, in all cases, we get

$$(u,v) \in \mathcal{L}_V \iff \begin{cases} t_n(uv) = t_n(u) \\ \overline{t}_{n-1}(uv) = \overline{t}_{n-1}(u) \\ t_n(vu) = t_n(v) \\ \overline{t}_{n-1}(vu) = \overline{t}_{n-1}(v) \end{cases}$$

$$\iff \begin{cases} t_n s(uv) = t_n s(u) \\ \sigma(uv) = \sigma(u) \\ \overline{t}_{n-1}(uv) = \overline{t}_{n-1}(u) \\ t_n s(vu) = t_n s(v) \\ \sigma(vu) = \sigma(v) \\ \overline{t}_{n-1}(vu) = \overline{t}_{n-1}(v) \end{cases} \qquad \text{by 2.2.9(i)}$$

$$\iff \begin{cases} c(u) = c(v) \\ \overline{t}_{n-1}(uv) = \overline{t}_{n-1}(u) \\ \overline{t}_{n-1}(vu) = \overline{t}_{n-1}(v) \end{cases}$$
$$\iff (u,v) \in \mathcal{L}_{V_r}.$$

(4) For $n = 3$ this is the case of $V_l = \mathbf{V}(R_3 = Q_3)$ and $V_r = \mathbf{V}(xa = axa)$. For $n > 3$, n odd, this is the case of $V_l = \mathbf{V}(R_n = Q_n)$ and $V_r = \mathbf{V}(R_{n-1} = S_{n-1})$. For n even, $V_l = \mathbf{V}(\overline{R}_n = \overline{Q}_n)$ and $V_r = \mathbf{V}(\overline{R}_{n-1} = \overline{S}_{n-1})$. In all cases, we have

$$(u,v) \in \mathcal{L}_V \iff \begin{cases} h_n(uv) = h_n(u) \\ \overline{i}_{n-1}(uv) = \overline{i}_{n-1}(u) \\ h_n(vu) = h_n(v) \\ \overline{i}_{n-1}(vu) = \overline{i}_{n-1}(v) \end{cases}$$

$$\iff \begin{cases} h_n s(uv) = h_n s(u) \\ \sigma(uv) = \sigma(u) \\ \overline{h}_{n-1}(uv) = \overline{h}_{n-1}(u) \\ \overline{i}_{n-1}(uv) = \overline{i}_{n-1}(u) \\ h_n s(vu) = h_n s(v) \\ \sigma(vu) = \sigma(v) \\ \overline{h}_{n-1}(vu) = \overline{h}_{n-1}(v) \\ \overline{i}_{n-1}(vu) = \overline{i}_{n-1}(v) \end{cases} \quad \text{by 2.2.9(i)}$$

$$\iff \begin{cases} \overline{h}_{n-1}(uv) = \overline{h}_{n-1}(u) \\ \overline{i}_{n-1}(uv) = \overline{i}_{n-1}(u) \\ \overline{h}_{n-1}(vu) = \overline{h}_{n-1}(v) \\ \overline{i}_{n-1}(vu) = \overline{i}_{n-1}(v) \end{cases}$$

$$\iff \begin{cases} \overline{i}_{n-1}(uv) = \overline{i}_{n-1}(u) \\ \overline{i}_{n-1}(vu) = \overline{i}_{n-1}(v) \end{cases} \quad \text{by 2.2.7(ii)}$$

$$\iff (u, v) \in \mathcal{L}_{V_r}.$$

Now, if $n = 3$, we get

$$(u, v) \in \mathcal{R}_V \iff \begin{cases} h_3(uv) = h_3(v) \\ \overline{i}_2(uv) = \overline{i}_2(v) \\ h_3(vu) = h_3(u) \\ \overline{i}_2(vu) = \overline{i}_2(u) \end{cases}$$

$$\iff \begin{cases} h_3(uv) = h_3(v) \\ h_3(vu) = h_3(u) \end{cases}$$

$$\iff (u, v) \in \mathcal{R}_{V_l}.$$

Finally, for $n > 3$,

$$(u, v) \in \mathcal{R}_V \iff \begin{cases} h_n(uv) = h_n(v) \\ \overline{i}_{n-1}(uv) = \overline{i}_{n-1}(v) \\ h_n(vu) = h_n(u) \\ \overline{i}_{n-1}(vu) = \overline{i}_{n-1}(u) \end{cases}$$

$$\iff \begin{cases} h_n(uv) = h_n(v) \\ \overline{i}_{n-1}\, e(uv) = \overline{i}_{n-1}\, e(v) \\ \varepsilon(uv) = \varepsilon(v) \\ i_{n-2}\,(uv) = i_{n-2}\,(v) \\ h_n(vu) = h_n(u) \\ \overline{i}_{n-1}\, e(vu) = \overline{i}_{n-1}\, e(u) \\ \varepsilon(vu) = \varepsilon(u) \\ i_{n-2}\,(vu) = i_{n-2}\,(u) \end{cases} \quad \text{by 2.2.9(iv)}$$

$$\iff \begin{cases} h_n(uv) = h_n(v) \\ i_{n-2}\,(uv) = i_{n-2}\,(v) \\ h_n(vu) = h_n(u) \\ i_{n-2}\,(vu) = i_{n-2}\,(u) \end{cases}$$

$$\iff \begin{cases} h_n(uv) = h_n(v) \\ h_n(vu) = h_n(u) \end{cases} \quad \text{by 2.2.7(i), 2.2.8(ii)}$$

$$\iff (u, v) \in \mathcal{R}_{V_i}.$$

Next we compute explicitly the cardinalities of the $\mathcal{J}$-classes of the free objects in the proper left varieties of $\mathbf{LB_0}$ and also the cardinalities of the $\mathcal{L}$-classes of those semigroups. By Theorem 3.1.6 and its corollary this is enough to deduce the cardinalities of the $\mathcal{J}$-classes of the free objects in the remainder varieties.

Proposition 3.1.10. *Let $k \geq 2$. Then*

(i) $c_k(xy = yx) = 1$.

(ii) $c_k(\overline{R}_2 = \overline{Q}_2) = k$.

(iii) $c_k(ax = axa) = k\ c_{k-1}(ax = axa) = k!$.

(iv) $c_k(R_3 = Q_3) = k^2 c_{k-1}(R_3 = Q_3) = (k!)^2$.

(v) $c_k(R_3 = S_3) = k^2 c_{k-1}(R_3 = S_3) c_{k-1}(ax = axa)$.

(vi) *For $n, m \geq 4$, n odd, m even and for $T \in \{Q, S\}$,*

$$c_k(R_n = T_n) = k^2 c_{k-1}(R_n = T_n) c_{k-1}(R_{n-1} = T_{n-1})$$

and

$$c_k(\overline{R}_m = \overline{T}_m) = k^2 c_{k-1}(\overline{R}_m = \overline{T}_m) c_{k-1}(\overline{R}_{m-1} = \overline{T}_{m-1}).$$

Proof. (i) This is clear. (See 2.2.10.)

(ii) Since

$$\mathbf{V}(\overline{R}_2 = \overline{Q}_2) = \mathbf{V}(ax = a) \vee \mathbf{V}(xy = yx),$$

we get

$$\begin{aligned}(u, v) \in \nu(\overline{R}_2 = \overline{Q}_2) &\iff (u, v) \in \nu(xy = yx) \cap \nu(ax = a) \\ &\iff \begin{cases} c(u) = c(v) \\ h_2(u) = h_2(v) \end{cases} \quad \text{by 2.2.10(i), (ii)}\end{aligned}$$

This means that two words are equivalent if and only if they have the same content and their first letters coincide. Consequently, $c_k(\overline{R}_2 = \overline{Q}_2) = k$.

(iii) If $u, v \in A^+$, then

$$(u, v) \in \nu(ax = axa) \iff \begin{cases} c(u) = c(v) \\ \sigma(u) = \sigma(v) \\ (s(u), s(v)) \in \nu(ax = axa) \end{cases}$$

Indeed,

$$\begin{aligned}(u, v) \in \nu(ax = axa) &\iff i_2(u) = i_2(v) \quad \text{by 2.2.10(iii)} \\ &\iff \begin{cases} c(u) = c(v) \\ \sigma(u) = \sigma(v) \\ i_2(s(u)) = i_2(s(v)) \end{cases} \quad \text{by 2.2.4}\end{aligned}$$

Thus, the number of elements in $U_X(ax = axa)$ is the number of pairs of the kind

$$(\sigma(u), [s(u)]_{\nu(ax=axa)}).$$

Since the words $s(u)$ have content of cardinality $k - 1$, we arrive at the recursion formula

$$c_k(ax = axa) = kc_{k-1}(ax = axa).$$

(iv) We have

$$
\begin{aligned}
(u,v)\in\nu(R_3=Q_3) &\iff h_3(u)=h_3(v) \quad \text{by 2.2.10(iv)}\\
&\iff \begin{cases} c(u)=c(v)\\ \sigma(u)=\sigma(v)\\ h_3(s(u))=h_3(s(v))\\ \overline{h}_2(u)=\overline{h}_2(v)\end{cases} \quad \text{by 2.2.9(i)}\\
&\iff \begin{cases} c(u)=c(v)\\ \sigma(u)=\sigma(v)\\ (s(u),s(v))\in\nu(R_3=Q_3)\\ h_2(\overline{u})=h_2(\overline{v})\end{cases}
\end{aligned}
$$

Thus, the number of elements in $U_X(R_3=Q_3)$ is the number of triples of the kind

$$(\sigma(u),h_2(\overline{u}),[s(u)]_{\nu(R_3=Q_3)}).$$

Since $h_2(\overline{u})$ is the last letter of u and since the words $s(u)$ have content of cardinality $k-1$, we arrive at the recursion formula

$$c_k(R_3=Q_3)=k^2c_{k-1}(R_3=Q_3).$$

(v) We have

$$
\begin{aligned}
(u,v)\in\nu(R_3=S_3) &\iff i_3(u)=i_3(v) \quad \text{by 2.2.10(iv)}\\
&\iff \begin{cases} c(u)=c(v)\\ \sigma(u)=\sigma(v)\\ i_3(s(u))=i_3(s(v))\\ \varepsilon(u)=\varepsilon(v)\\ \overline{i}_2(e(u))=\overline{i}_2(e(v))\end{cases} \quad \text{by 2.2.9(iii)}
\end{aligned}
$$

Therefore, the number of elements in $U_X(R_3=S_3)$ is the number of quadruples of the kind

$$(\sigma(u),\varepsilon(u),[s(u)]_{\nu(R_3=S_3)},[e(u)]_{\nu(xa=axa)}).$$

Since the elements $s(u)$ and $e(u)$ have contents of cardinality $k-1$, we arrive at the recursion formula

$$c_k(R_3 = S_3) = k^2 c_{k-1}(R_3 = S_3) c_{k-1}(xa = axa).$$

(vi) Let $n \geq 4$, n odd and let $T \in \{Q, S\}$. By 2.2.10 we get

$$(u,v) \in \nu(R_n = T_n) \quad \Longleftrightarrow \quad t_n(u) = t_n(v)$$

where $t = h$ if $T = Q$ and $t = i$ if $T = S$. By 2.2.9(iii) we deduce

$$(u,v) \in \nu(R_n = T_n) \iff \begin{cases} c(u) = c(v) \\ \sigma(u) = \sigma(v) \\ \varepsilon(u) = \varepsilon(v) \\ t_n(s(u)) = t_n(s(v)) \\ \overline{t}_{n-1}(e(u)) = \overline{t}_{n-1}(e(v)) \end{cases}$$

Therefore, the number of elements in $U_X(R_n = T_n)$ is the number of quadruples of the kind

$$(\sigma(u), \varepsilon(u), [s(u)]_{\nu(R_n=T_n)}, [e(u)]_{\nu(R_{n-1}=T_{n-1})})$$

and we arrive at the recursion formula

$$c_k(R_n = T_n) = k^2 c_{k-1}(R_n = T_n) c_{k-1}(R_{n-1} = T_{n-1}).$$

The proof of the second statement is similar to this one, now using 2.2.9(iv).

Proposition 3.1.11. *Let* $k \geq 2$. *Then*

(i) $L_k(xy = yx) = R_k(xy = yx) = 1 = c_k(xy = yx)$.

(ii) $R_k(\overline{R}_2 = \overline{Q}_2) = 1$

and

$L_k(\overline{R}_2 = \overline{Q}_2) = k = c_k(\overline{R}_2 = \overline{Q}_2)$.

(iii) $R_k(ax = axa) = 1$

and

$$L_k(ax = axa) = k! = c_k(ax = axa) = kc_{k-1}(ax = axa).$$

(iv) $R_k(R_3 = Q_3) = k$

and

$$L_k(R_3 = Q_3) = kc_{k-1}(R_3 = Q_3).$$

(v) $R_k(R_3 = S_3) = k! = c_k(ax = axa)$

and

$$L_k(R_3 = S_3) = kc_{k-1}(R_3 = S_3).$$

(vi) *For* $n \geq 4$ *and* $T \in \{Q, S\}$

$$R_k(R_n = T_n) = kc_{k-1}(R_{n-1} = T_{n-1}) \text{ , for } n \text{ odd,}$$

$$R_k(\overline{R}_n = \overline{T}_n) = kc_{k-1}(\overline{R}_{n-1} = \overline{T}_{n-1}) \text{ , for } n \text{ even}$$

and

$$L_k(R_n = T_n) = kc_{k-1}(R_n = T_n) \text{ , for } n \text{ odd,}$$

$$L_k(\overline{R}_n = \overline{T}_n) = kc_{k-1}(\overline{R}_n = \overline{T}_n) \text{ , for } n \text{ even.}$$

Proof. (i), (ii) and (iii) follow immediately from 3.1.8 and 3.1.10(i)–(iii).

(iv) We have

$$\begin{aligned} \mathcal{L}\mathrm{v}_{(R_3=Q_3)} &= \mathcal{L}\mathrm{v}_{(R_2=Q_2)} && \text{by 3.1.6} \\ &= \mathrm{id}_{F_A(R_2=Q_2)} && \text{by 3.1.8(i).} \end{aligned}$$

Therefore,

$$R_k(R_3 = Q_3) = c_k(R_2 = Q_2) = k$$

and

$$L_k(R_3 = Q_3) = \frac{c_k(R_3 = Q_3)}{k}$$

$$= \frac{k^2 c_{k-1}(R_3 = Q_3)}{k} \quad \text{by 3.1.10(iv)}$$

$$= kc_{k-1}(R_3 = Q_3).$$

(v) We have

$$\mathcal{L}_{\mathbf{V}(R_3=S_3)} = \mathcal{L}_{\mathbf{V}(xa=axa)} \quad \text{by 3.1.6}$$
$$= \mathrm{id}_{F_A(xa=axa)} \quad \text{by 3.1.8(i).}$$

Hence,

$$R_k(R_3 = S_3) = c_k(xa = axa)$$

and

$$L_k(R_3 = S_3) = \frac{c_k(R_3 = S_3)}{c_k(xa = axa)}$$

$$= \frac{k^2 c_{k-1}(R_3 = S_3) c_{k-1}(xa = axa)}{kc_{k-1}(xa = axa)} \quad \text{by 3.1.10(iii),(v)}$$

$$= kc_{k-1}(R_3 = S_3).$$

(vi) Let $n \geq 4$. For $n = 4$ we get

$$\mathcal{L}_{\mathbf{V}(\overline{R}_4=\overline{T}_4)} = \mathcal{L}_{\mathbf{V}(\overline{R}_3=\overline{T}_3)} \quad \text{by 3.1.6.}$$

Hence,

$$\begin{aligned} R_k(\overline{R}_4 = \overline{T}_4) &= R_k(\overline{R}_3 = \overline{T}_3) \\ &= L_k(R_3 = T_3) \\ &= kc_{k-1}(R_3 = T_3) \quad \text{by (iv),(v)} \\ &= kc_{k-1}(\overline{R}_3 = \overline{T}_3) \end{aligned}$$

and

$$\begin{aligned} L_k(\overline{R}_4 = \overline{T}_4) &= \frac{c_k(\overline{R}_4 = \overline{T}_4)}{R_k(\overline{R}_4 = \overline{T}_4)} \\ &= \frac{k^2 c_{k-1}(\overline{R}_4 = \overline{T}_4) c_{k-1}(\overline{R}_3 = \overline{T}_3)}{kc_{k-1}(\overline{R}_3 = \overline{T}_3)} \quad \text{by 3.1.10(vi)} \\ &= kc_{k-1}(\overline{R}_4 = \overline{T}_4). \end{aligned}$$

For $n = 5$ we get

$$\mathcal{L}_{\mathrm{V}(R_5=T_5)} = \mathcal{L}_{\mathrm{V}(R_4=T_4)} \quad \text{by 3.1.6.}$$

Hence,

$$\begin{aligned} R_k(R_5 = T_5) &= R_k(R_4 = T_4) \\ &= L_k(\overline{R}_4 = \overline{T}_4) \\ &= kc_{k-1}(\overline{R}_4 = \overline{T}_4) \\ &= kc_{k-1}(R_4 = T_4) \end{aligned}$$

and

$$L_k(R_5 = T_5) = \frac{c_k(R_5 = T_5)}{R_k(R_5 = T_5)}$$

$$= \frac{k^2 c_{k-1}(R_5 = T_5) c_{k-1}(R_4 = T_4)}{k c_{k-1}(R_4 = T_4)} \quad \text{by 3.1.10(vi)}$$

$$= k c_{k-1}(R_5 = T_5).$$

Let $n > 4$ and suppose the result is valid for $n-1$. For n odd, we get

$$\mathcal{L}_{\mathbf{V}(R_n = T_n)} = \mathcal{L}_{\mathbf{V}(R_{n-1} = T_{n-1})} \quad \text{by 3.1.6}$$

Hence,

$$\begin{aligned} R_k(R_n = T_n) &= R_k(R_{n-1} = T_{n-1}) \\ &= L_k(\overline{R}_{n-1} = \overline{T}_{n-1}) \\ &= k c_{k-1}(\overline{R}_{n-1} = \overline{T}_{n-1}) \\ &= k c_{k-1}(R_{n-1} = T_{n-1}), \end{aligned}$$

by the induction hypothesis. Then,

$$L_k(R_n = T_n) = \frac{c_k(R_n = T_n)}{R_k(R_n = T_n)}$$

$$= \frac{k^2 c_{k-1}(R_n = T_n) c_{k-1}(R_{n-1} = T_{n-1})}{k c_{k-1}(R_{n-1} = T_{n-1})} \quad \text{by 3.1.10(vi)}$$

$$= k c_{k-1}(R_n = T_n).$$

For n even, the proof is similar.

Example 3.1.12. Let $V = \mathbf{V}(R_3 = S_3)$. By 3.1.10(v) we have that

$$c_2(R_3 = S_3) = 4 c_1(R_3 = S_3) c_1(ax = axa) = 4$$

$$
\begin{aligned}
c_3(R_3 = S_3) &= 9c_2(R_3 = S_3)c_2(ax = axa) \\
&= 9 \times 4 \times 2 \quad \text{by 3.1.6(iii)} \\
&= 72
\end{aligned}
$$

If $|A| = 2$ by 3.1.1 we get

$$
\begin{aligned}
|F_A(V)| &= \sum_{k=1}^{2} \binom{2}{k} c_k(V) \\
&= \binom{2}{1} c_1(V) + \binom{2}{2} c_2(V) \\
&= 2 + 4 \\
&= 6.
\end{aligned}
$$

and if $|A| = 3$ we get

$$
\begin{aligned}
|F_A(V)| &= \sum_{k=1}^{3} \binom{3}{k} c_k(V) \\
&= \binom{3}{1} c_1(V) + \binom{3}{2} c_2(V) + \binom{3}{3} c_3(V) \\
&= 3 + 3 \times 4 + 72 \\
&= 87.
\end{aligned}
$$

We finish this section with two examples, where we compute explicitly the free semigroups generated by a certain A. We use an algorithm similar to the one used by Pin in [**20**, chapter 1.3].

Example 3.1.13. Let $V = \mathbf{V}(axa = ax)$. By 2.2.14(i), $F_A(V) \simeq i_2(A^+)$.

Let $A = \{a, b\}$. Since $i_2(u)$ ($u \in A^+$) is the word obtained by writing the variables of u in order of first occurrence, we get

u	$i_2(u)$
a	a
b	b
ba	ba
ab	ab
aba	ab
bab	ba

Hence, $i_2(A^+) = \{a, b, ab, ba\}$.

Let $A = \{a, b, c\}$. Since

$$|F_A(V)| = \sum_{k=1}^{3} \binom{3}{k} c_k(V) = 3 + 3 \times 2 + 3! = 15$$

we can reduce drastically the number of cases that have to be considered and we get

u	$i_2(u)$
a	a
b	b
c	c
ba	ba
ca	ca
ab	ab
cb	cb
ac	ac
bc	bc
aba	ab
cba	cba
aca	ac
bca	bca
bab	ba
cab	cab
acb	acb
bcb	bc
bac	bac
cac	ca
abc	abc

Thus

$$i_2(A^+) = \{a, b, c, ba, ca, ab, cb, ac, bc, cba, bca, cab, acb, bac, abc\}.$$

Example 3.1.14. Let $V = \mathbf{V}(R_3 = S_3)$. For $A = \{a, b\}$ we have

u	$i_3(u)$
a	a^2
b	b^2
ba	b^2a^2b
ab	a^2b^2a
aba	a^2bab
bab	b^2aba

since

$$i_3(a) = i_3(s(a))\sigma(a)\overline{i}_2(a) = a^2,$$

$$\begin{aligned} i_3(ba) &= i_3(s(ba))\sigma(ba)\overline{i}_2(ba) \\ &= i_3(b)aab \\ &= b^2a^2b \end{aligned}$$

and

$$\begin{aligned} i_3(aba) &= i_3(s(aba))\sigma(aba)\overline{i}_2(aba) \\ &= i_3(a)bab \\ &= a^2bab. \end{aligned}$$

Hence

$$i_3(A^+) = \{a^2, b^2, b^2a^2b, a^2bab, b^2aba\}.$$

2. Collapsing of the finitely generated free objects

In this section the alphabet A is finite.

We analyse the following question: given two varieties of bands V and W such that $V \subseteq W$, for which cardinalities of A do we have $F_A(V) = F_A(W)$?

We give an example.

Example 3.2.1. Let $|A| = 3$. By 3.1.10 we get

$$c_2(R_4 = S_4) = 4c_1(R_4 = S_4)c_1(R_3 = S_3) = 4,$$

$$c_3(R_4 = S_4) = 9c_2(R_4 = S_4)c_2(R_3 = S_3)$$

$$= 9 \times 4 \times 4, \qquad \text{by 3.1.12.}$$

$$c_2(R_4 = Q_4) = 4c_1(R_4 = Q_4)c_1(R_3 = Q_3)$$

$$= c_2(R_4 = S_4)$$

and

$$c_3(R_4 = Q_4) = 9c_2(R_4 = Q_4)c_2(R_3 = Q_3)$$

$$= 9 \times 4 \times 4$$

$$= c_3(R_4 = S_4)$$

Hence, $|F_A(R_4 = S_4)| = |F_A(R_4 = Q_4)|$, by 3.1.1. Since $\mathbf{V}(R_4 = Q_4) \subseteq \mathbf{V}(R_4 = S_4)$, 2.1.20 yields that there is a surjective morphism $f : F_A(R_4 = S_4) \to F_A(R_4 = Q_4)$. Hence f is a bijection and $F_A(R_4 = S_4) \simeq F_A(R_4 = Q_4)$.

Suppose now that $|A| = 4$. Again by 3.1.10 we get

$$c_4(R_4 = S_4) = 16c_3(R_4 = S_4)c_3(R_3 = S_3)$$

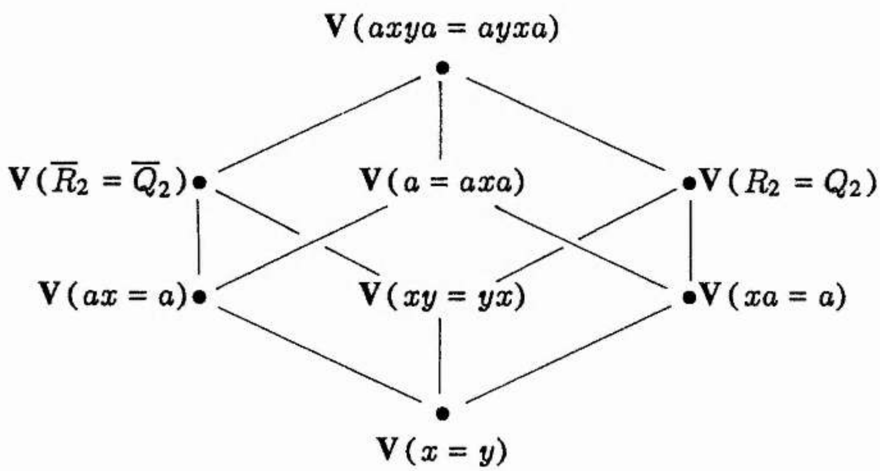
$$= 16 \times 144 \times 72 \qquad \text{by 3.1.12}$$

and

$$
\begin{aligned}
c_4(R_4 = Q_4) &= 16\, c_3(R_4 = Q_4)\, c_3(R_3 = Q_3) \\
&= 16 \times 144 \times 36 \\
&< c_4(R_4 = S_4)
\end{aligned}
$$

Hence, $|\ F_A(R_4 = Q_4)\ | < |\ F_A(R_4 = S_4)\ |$.

Let us consider the following sublattice of **LB**



Proposition 3.2.2. (i) *For all* $V \in \mathbf{LB}$,

$$|A| = 1 \iff F_A(V) = F_A(x = y).$$

(ii) *If* $|A| \geq 2$, *the semigroups* $F_A(ax = a), F_A(xa = a), F_A(axa = a)$ *are distinct. They are also distinct from any free object in a variety of* $\mathbf{LB_0}$.

(iii) *If* $|A| \geq 2$ *and* $V \in \mathbf{LB_0}$, $V \neq \mathbf{V}(xy = yx)$, *then*

$$F_A(V) \neq F_A(xy = yx).$$

Proof. (i) This is clearly true.

(ii) In the previous paragraph we saw that

$$|F_A(ax = a)| = |A|$$

and

$$|F_A(axa = a)| = |A|^2.$$

Hence, for $|A| \geq 2$

$$F_A(ax = a) \not\cong F_A(axa = a).$$

Clearly, for $|A| \geq 2$

$$F_A(ax = a) \not\cong F_A(xa = a),$$

since $F_A(ax = a)$ is a left zero band and $F_A(xa = a)$ is a right zero band.

Also, from 3.1.1 and 3.1.10 we deduce that

$$|F_A(R_2 = Q_2)| = \sum_{k=1}^{|A|} \binom{|A|}{k} k$$

and

$$\begin{aligned} |F_A(axya = ayxa)| &= \sum_{k=1}^{|A|} \binom{|A|}{k} c_k(axya = ayxa) \\ &= \sum_{k=1}^{|A|} \binom{|A|}{k} L_k(\overline{R}_2 = \overline{Q}_2) R_k(R_2 = Q_2) \quad \text{by 3.1.6} \\ &= \sum_{k=1}^{|A|} \binom{|A|}{k} k^2 \quad \text{by 3.1.11.} \end{aligned}$$

Therefore, for $|A| \geq 2$

$$|F_A(ax = a)| < |F_A(R_2 = Q_2)|$$

and

$$|F_A(axa = a)| < |F_A(axya = ayxa)|.$$

Hence, for all $V \in \mathbf{LB_0}$

$$F_A(ax = a), F_A(axa = a) \neq F_A(V).$$

(iii) is also clear, by 3.1.1 and 3.1.10.

We compare next the cardinalities of the $\mathcal{L}$ $[\mathcal{R}]$-classes of the $\mathcal{J}$-classes $U_X(V)$, where $V \in \mathbf{LB_0}$. In general, we have

Proposition 3.2.3. *Let $V \subseteq W$ be varieties of $\mathbf{LB_0}$ and let $k \geq 1$. Then*

$$R_k(V) \leq R_k(W)$$

and

$$L_k(V) \leq L_k(W).$$

Proof. Let $u, v \in A^+$ be such that $(u, v) \notin \mathcal{L}_V$. Then $(u, v) \notin \mathcal{L}_W$. Hence, the number of $\mathcal{L}$ $[\mathcal{R}]$-classes in a $\mathcal{J}$-class $U_X(W)$ is bigger or equal to the number of $\mathcal{L}$ $[\mathcal{R}]$-classes in the corresponding $\mathcal{J}$-class $U_X(V)$; that is, the cardinality $R_k(W)$ $[L_k(W)]$ of an $\mathcal{R}$ $[\mathcal{L}]$-class of $U_X(W)$ is bigger or equal to the cardinality $R_k(V)$ $[L_k(V)]$ of an $\mathcal{R}$ $[\mathcal{L}]$-class of $U_X(V)$. (See 3.1.4.)

In the next lemma we compare the cardinalities of the $\mathcal{J}$-classes of the free objects in the proper left varieties.

Lemma 3.2.4. *Let* $k \geq 2$.

(i) $c_2(R_2 = Q_2) = c_2(ax = axa) < c_2(R_3 = Q_3)$

and for $k \geq 3$,

$$c_k(R_2 = Q_2) < c_k(ax = axa) < c_k(R_3 = Q_3).$$

(ii) *Let* $n \geq 3$ *and let* $n > k$. *Then*

$$c_k(R_n = Q_n) = c_k(R_n = S_n) = c_k(R_{k+1} = Q_{k+1})$$

and

$$c_k(R_n = S_n) = c_k(R_{n+1} = Q_{n+1}) = c_k(R_{k+1} = S_{k+1}).$$

(iii) *If* $3 \leq n \leq k$

$$c_k(R_n = Q_n) < c_k(R_n = S_n) < c_k(R_{n+1} = Q_{n+1}).$$

Proof.(i) The statement is clear since

$$c_k(R_2 = Q_2) = k,$$

$$c_k(ax = axa) = k!$$

and

$$c_k(R_3 = Q_3) = (k!)^2.$$

(ii) By 3.1.3, it is enough to show that if $n > k$, then

$$c_k(R_n = Q_n) = c_k(R_{k+1} = Q_{k+1}).$$

The proof is by induction on $k \geq 2$ and will make use of the formulas established in 3.1.10. For $k = 2$ the statement is obvious for $n = 3$. If $n \geq 4$, we have

$$\begin{aligned} c_2(R_n = Q_n) &= 4c_1(R_n = Q_n)c_1(R_{n-1} = Q_{n-1}) \\ &= 4 \\ &= 4c_1^2(R_3 = Q_3) \\ &= c_2(R_3 = Q_3). \end{aligned}$$

Let $k \geq 2$ and $n > k + 1$. By the induction hypothesis

$$\begin{aligned} c_{k+1}(R_n = Q_n) &= (k+1)^2 c_k(R_n = Q_n)c_k(R_{n-1} = Q_{n-1}) \\ &= (k+1)^2 c_k(R_{k+1} = Q_{k+1})c_k(R_{k+1} = Q_{k+1}) \\ &= (k+1)^2 c_k(R_{k+2} = Q_{k+2})c_k(R_{k+1} = Q_{k+1}) \\ &= c_{k+1}(R_{k+2} = Q_{k+2}). \end{aligned}$$

The proof of the second statement is analogous with this one.

(iii) The proof is by induction on $k \geq 3$ and we refer again to 3.1.10. For $k = 3$ we get

$$\begin{aligned} c_3(R_3 = Q_3) &= (3!)^2 = 9 \times 4 \\ c_3(R_3 = S_3) &= 9c_2(R_3 = S_3)c_2(ax = axa) \\ &= 9 \times 4 \times 2 \\ &> 9 \times 4 \end{aligned}$$

and

$$\begin{aligned} c_3(R_4 = Q_4) &= 9c_2(R_4 = Q_4)c_2(R_3 = Q_3) \\ &= 9 \times 4 \times 4 \\ &> 9 \times 4 \times 2. \end{aligned}$$

Let $k \geq 3$ and $3 \leq n \leq k+1$. For $n = 3$ we get

$$c_{k+1}(R_3 = Q_3) = (k+1)^2 c_k(R_3 = Q_3),$$

$$\begin{aligned} c_{k+1}(R_3 = S_3) &= (k+1)^2 c_k(R_3 = S_3) c_k(ax = axa) \\ &> c_{k+1}(R_3 = Q_3), \end{aligned}$$

since $c_k(ax = axa) > 1$ and $c_k(R_3 = S_3) \geq c_k(R_3 = Q_3)$. Finally,

$$\begin{aligned} c_{k+1}(R_4 = Q_4) &= (k+1)^2 c_k(R_4 = Q_4) c_k(R_3 = Q_3) \\ &> c_{k+1}(R_3 = S_3), \end{aligned}$$

since $c_k(R_4 = Q_4) \geq c_k(R_3 = S_3)$ and $c_k(R_3 = Q_3) > c_k(ax = axa)$ (by (i)).

If $n \geq 4$

$$c_{k+1}(R_n = Q_n) = (k+1)^2 c_k(R_n = Q_n) c_k(R_{n-1} = Q_{n-1}),$$

$$c_{k+1}(R_n = S_n) = (k+1)^2 c_k(R_n = S_n) c_k(R_{n-1} = S_{n-1})$$

and

$$c_{k+1}(R_{n+1} = Q_{n+1}) = (k+1)^2 c_k(R_{n+1} = Q_{n+1}) c_k(R_n = Q_n).$$

Since

$$c_k(R_n = Q_n) \leq c_k(R_n = S_n) \leq c_k(R_{n+1} = Q_{n+1})$$

and since, by the induction hypothesis,

$$c_k(R_{n-1} = Q_{n-1}) < c_k(R_{n-1} = S_{n-1}) < c_k(R_n = Q_n),$$

we deduce that

$$c_{k+1}(R_n = Q_n) < c_{k+1}(R_n = S_n) < c_{k+1}(R_{n+1} = Q_{n+1}),$$

as required.

Proposition 3.2.5. (i) *For $l, l' \geq 3$, $l \neq l'$, we have*

$$F_A(R_l = S_l) = F_A(R_{l'} = S_{l'}) \iff l, l' > |A|.$$

(ii) *For $l, l' \geq 2$, $l \neq l'$, we have*

$$F_A(R_l = Q_l) = F_A(R_{l'} = Q_{l'}) \iff l, l' > |A|.$$

(iii) *For $l \geq 3$, $l' \geq 2$, $l \neq l'$, we have*

$$F_A(R_l = S_l) = F_A(R_{l'} = Q_{l'}) \iff l, l' > |A|.$$

Proof. (i) By the considerations made in the previous sections (see 2.1.20 and 3.1.1), it is enough to show that

$$|F_A(R_l = S_l)| = |F_A(R_{l'} = S_{l'})| \iff l, l' > |A|.$$

By 3.1.1 we have

$$|F_A(R_l = S_l)| = \sum_{k=1}^{|A|} \binom{|A|}{k} c_k(R_l = S_l)$$

and

$$|F_A(R_{l'} = S_{l'})| = \sum_{k=1}^{|A|} \binom{|A|}{k} c_k(R_{l'} = S_{l'}).$$

If $l, l' > |A|$, then, by 3.2.4(ii),

$$c_k(R_l = S_l) = c_k(R_{l'} = S_{l'}),$$

for all $k \in \{1, \dots, |A|\}$. Therefore,

$$|F_A(R_l = S_l)| = |F_A(R_{l'} = S_{l'})|.$$

Conversely, suppose that $l, l' > |A|$. Now

$$F_A(R_l = S_l) \subseteq F_A(R_{l'} = S_{l'})$$

or

$$F_A(R_{l'} = S_{l'}) \subseteq F_A(R_l = S_l).$$

Suppose the first inclusion occurs. By 3.1.3

$$c_k(R_l = S_l) \leq c_k(R_{l'} = S_{l'}),$$

for all $1 \leq k \leq |A|$. Hence

$$c_k(R_l = S_l) = c_k(R_{l'} = S_{l'}),$$

for all $1 \leq k \leq |A|$, in particular for $k = |A|$. Again by 3.2.4, we deduce that $l > |A|, l' > |A|$.

If the second inclusion occurs the reasoning is similar.

(ii) If $l = 2, l' = 3$, 3.2.4 yields

$$|F_A(R_2 = Q_2)| = |F_A(R_3 = Q_3)| \iff |A| = 1, \tag{3.2.5.1}$$

since $c_k(R_2 = Q_2) < c_k(R_3 = Q_3)$, for all $2 \leq k \leq |A|$. Hence

$$|F_A(R_2 = Q_2)| = |F_A(R_3 = Q_3)| \iff |A| < 2, 3.$$

For $l, l' \geq 3$, the proof is analogous with the proof of (i).

(iii) For $l = 3$ and $l' = 2$, by (3.2.5.1), we get

$$|F_A(R_3 = S_3)| = |F_A(R_2 = Q_2)| \iff |A| < 2, 3.$$

Suppose that $l, l' \geq 3$ and that $l, l' > |A|$. For $l' < l$, 3.2.4 yields

$$c_k(R_{l'} = Q_{l'}) = c_k(R_{l'} = S_{l'}) = c_k(R_l = S_l)$$

for all $1 \leq k \leq |A|$. Thus

$$|F_A(R_{l'} = Q_{l'})| = |F_A(R_l = S_l)|.$$

For $l < l'$, again by 3.2.4, we have

$$c_k(R_l = S_l) = c_k(R_{l+1} = Q_{l+1}) = c_k(R_{l'} = Q_{l'}).$$

Conversely, suppose that

$$F_A(R_{l'} = Q_{l'}) = F_A(R_l = S_l).$$

We have

$$F_A(R_{l'} = Q_{l'}) \subseteq F_A(R_l = S_l) \quad (\text{if } l' \leq l)$$

or

$$F_A(R_l = S_l) \subseteq F_A(R_{l'} = Q_{l'}) \quad (\text{if } l' > l).$$

Suppose the first inclusion occurs. Then

$$c_k(R_{l'} = Q_{l'}) \leq c_k(R_l = S_l),$$

for all $1 \leq k \leq |A|$, in particular for $k = |A|$. The equality of the cardinalities of the free objects gives that

$$c_{|A|}(R_{l'} = Q_{l'}) = c_{|A|}(R_l = S_l)$$

Again by 3.2.4, we deduce that $l, l' < |A|$.

Corollary 3.2.6. *Let* $T \in \{Q, S\}$, $l \geq 3$ *and let* $U \in \mathbf{V}(R_l = T_l)$. *If* U *is generated by* $l-2$ *elements, then* $U \in \mathbf{V}(R_{l-1} = T_{l-1})$.

Proof. If U is generated by $l-2$ elements, then for any A such that $|A| = l-2$ there is a map $\alpha : A \to U$ such that the morphism from $F_A(R_l = T_l)$ into U that makes diagram (2.1.17.1) commutative is an epimorphism.

On the other hand, 3.2.5(i),(ii) yields that

$$F_A(R_l = T_l) = F_A(R_{l-1} = T_{l-1}).$$

Hence $F_A(R_l = T_l) \in \mathbf{V}(R_{l-1} = T_{l-1})$ and therefore $U \in \mathbf{V}(R_{l-1} = T_{l-1})$.

The next statement is the equivalent, for $\mathcal{L}$-classes, to Lemma 3.2.4.

Lemma 3.2.7. *Let* $n, k \geq 2$. *Then*

(i) $L_2(\overline{R}_2 = \overline{Q}_2) = L_2(ax = axa) = L_2(R_3 = Q_3)$

and for $k \geq 3$,

$$L_k(\overline{R}_2 = \overline{Q}_2) < L_k(ax = axa) < L_k(R_3 = Q_3).$$

(ii) *If* $n \geq k, n \geq 3$

$$L_k(R_n = Q_n) = L_k(R_n = S_n) = L_k(\overline{R}_{n+1} = \overline{Q}_{n+1}), \text{ for } n \text{ odd}$$

and

$$L_k(\overline{R}_n = \overline{Q}_n) = L_k(\overline{R}_n = \overline{S}_n) = L_k(R_{n+1} = Q_{n+1}), \text{ for } n \text{ even.}$$

(iii) *If* $3 \leq n < k$,

$$L_k(R_n = Q_n) < L_k(R_n = S_n) < L_k(\overline{R}_{n+1} = \overline{Q}_{n+1}), \; for\ n\ odd$$

and

$$L_k(\overline{R}_n = \overline{Q}_n) < L_k(\overline{R}_n = \overline{S}_n) < L_k(R_{n+1} = Q_{n+1}), \; for\ n\ even.$$

Proof.(i) The statement is clearly true by 3.1.11, since

$$L_k(\overline{R}_2 = \overline{Q}_2) = k,$$

$$L_k(ax = axa) = k!$$

and

$$L_k(R_3 = Q_3) = k\big((k-1)!\big)^2.$$

(ii) Let $n \geq k$, n odd. By 3.1.11,

$$L_k(R_n = Q_n) = kc_{k-1}(R_n = Q_n),$$

$$L_k(R_n = S_n) = kc_{k-1}(R_n = S_n)$$

and

$$L_k(\overline{R}_{n+1} = \overline{Q}_{n+1}) = kc_{k-1}(\overline{R}_{n+1} = \overline{Q}_{n+1}).$$

and the statement follows from the fact that

$$c_{k-1}(R_n = Q_n) = c_{k-1}(R_n = S_n) = c_{k-1}(\overline{R}_{n+1} = \overline{Q}_{n+1}). \quad \text{(See 3.2.4(ii).)}$$

(iii) The reasoning is analogous with that used in the proof of (ii), but now using 3.2.4(iii).

We finish this study with a final result, which gives a tool to check whenever two free objects in two related varieties of $\mathbf{LB_0}$ coincide.

Theorem 3.2.8. *Let $V, W \in \mathbf{LB_0}$ be proper varieties such that $V \subseteq W$. Then*

$$F_A(V) = F_A(W) \iff \begin{cases} L_{|A|}(V_l) = L_{|A|}(W_l) \\ R_{|A|}(V_r) = R_{|A|}(W_r). \end{cases}$$

The proof of this theorem depends on two lemmas which we now state.

Lemma 3.2.9. *Let $V, W \in \mathbf{LB_0}$ be proper varieties such that $V \subseteq W$. Then, for all $k \geq 1$*

$$c_k(V) = c_k(W) \iff \begin{cases} L_k(V_l) = L_k(W_l) \\ R_k(V_r) = R_k(W_r). \end{cases}$$

Proof. By 3.1.7 we have

$$c_k(V) = L_k(V_l) R_k(V_r)$$

and

$$c_k(W) = L_k(W_l) R_k(W_r).$$

Consequently, if $L_k(V_l) = L_k(W_l)$ and $R_k(V_r) = R_k(W_r)$ we deduce that $c_k(V) = c_k(W)$.

The converse follows from de fact that

$$L_k(V_l) \leq L_k(W_l)$$

and

$$R_k(V_r) \leq R_k(W_r),$$

since $V \subseteq W$. (See 3.2.3.)

Lemma 3.2.10. *Let $V, W \in \mathbf{LB_0}$ be proper varieties. Let $k \geq 2$ and $k' < k$.*

(i) *If V and W are left varieties, then*

$$L_k(V) = L_k(W) \Longrightarrow L_{k'}(V) = L_{k'}(W).$$

(ii) *If V and W are right varieties, then*

$$R_k(V) = R_k(W) \Longrightarrow R_{k'}(V) = R_{k'}(W).$$

This lemma results immediately from Lemma 3.2.7.

We now prove the theorem.

Proof of Theorem 3.2.8. As we noticed before

$$F_A(V) = F_A(W) \Longleftrightarrow |F_A(V)| = |F_A(W)|.$$

If $L_{|A|}(V_l) = L_{|A|}(W_l)$ and $R_{|A|}(V_r) = R_{|A|}(W_r)$, Lemma 3.2.10 yields that, for all $1 \leq k \leq |A|$

$$L_k(V_l) = L_k(W_l) \quad \text{and} \quad R_k(V_r) = R_k(W_r).$$

Hence, Lemma 3.2.9 yields that

$$c_k(V) = c_k(W) \quad (1 \leq k \leq |A|).$$

Hence, $F_A(V) = F_A(W)$.

Conversely, suppose that $F_A(V) = F_A(W)$. Since $V \subseteq W$, by 3.1.3 we know that

$$c_k(V) \leq c_k(W) \quad (1 \leq k \leq |A|).$$

Thus, according with 3.1.1, we deduce

$$c_k(V) = c_k(W) \quad (1 \leq k \leq |A|).$$

In particular, we get that $c_{|A|}(V) = c_{|A|}(W)$ and Lemma 3.2.9 yields again

$$L_{|A|}(V_l) = L_{|A|}(W_l) \quad \text{and} \quad R_{|A|}(V_r) = R_{|A|}(W_r),$$

as required.

3. Varieties of band monoids

The results for varieties of **LBM** are easily deducible from the results for varieties of **LB**. Indeed, if $V \in \mathbf{LBM}$ and $< V >_\mathbf{S}$ is the Birkhoff variety of bands generated by V (see 2.1.2) we have

Theorem 3.3.1. *If* $V \in \mathbf{LBM_0}$, *then*

$$F_A(V) = (F_A(< V >_\mathbf{S}))^\mathbf{I}$$

Proof. Let $V \in \mathbf{LBM_0}$ and let $M \in V$. Since $M \in < V >_\mathbf{S}$, there is a unique morphism $\psi : F_A(< V >_\mathbf{S}) \to M$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "\iota_{<V>_S}"] \arrow[d, "\varphi"'] & F_A(< V >_\mathbf{S}) \arrow[dl, "\psi"] \\ M & \end{tikzcd} \qquad (3.3.1.1)$$

Define $\psi^{\mathrm{I}} : (F_A(< V >_{\mathrm{S}}))^{\mathrm{I}} \to M$ by

$$\psi^{\mathrm{I}} = \begin{cases} \psi(x), & \text{if } x \in F_A(< V >_{\mathrm{S}}); \\ \mathrm{id}_M, & \text{if } x = 1. \end{cases}$$

Let $\mathrm{i} : F_A(< V >_{\mathrm{S}}) \to (F_A(< V >_{\mathrm{S}}))^{\mathrm{I}}$ be the inclusion map. Then ψ^{I} is the unique homomorphism from $(F_A(< V >_{\mathrm{S}}))^{\mathrm{I}}$ into M such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "\mathrm{i} \circ]_{<V>_{\mathrm{S}}}"] \arrow[d, "\varphi"'] & (F_A(< V >_{\mathrm{S}}))^{\mathrm{I}} \arrow[dl, "\psi^{\mathrm{I}}"] \\ M & \end{tikzcd}$$

It only remains to prove that $(F_A(< V >_{\mathrm{S}}))^{\mathrm{I}}$ is in V. This is a consequence of the following lemma:

Lemma 3.3.2. *Let W be a variety of bands such that $W \in \chi(\mathbf{LBM_\emptyset})$. Then for any semigroup S*

$$S \in W \Rightarrow S^{\mathrm{I}} \in W.$$

Proof. We notice that if S is a band, S^{I} is also a band.

It is enough to prove the result for the varieties $W = \chi(V)$ such that V is a strictly left or a strictly right variety of $\mathbf{LBM_\emptyset}$.

Indeed, given a proper variety of band monoids V, by 2.3.2, $\chi(V) = \chi(V^l) \cap \chi(V^r)$. Therefore if $S \in \chi(V)$, then $S^{\mathrm{I}} \in \chi(V^l)$ and $S^{\mathrm{I}} \in \chi(V^r)$. Thus $S^{\mathrm{I}} \in \chi(V)$.

Let $S \in \mathbf{V}(ax = axa)$ and let $s, t \in S^{\mathrm{I}}$. If $s = 1$ the substitution $a = 1, x = t$ results in $t = t$ and the substitution $a = t, x = 1$ results in $t^2 = t$. Hence $S^{\mathrm{I}} \in \mathbf{V}(ax = axa)$.

Let $S \in \mathbf{V}(R_3 = S_3)$. Then S satisfies the identity $R_3 = S_3$, that is, S satisfies the identity

$$x_1 x_2 x_3 = x_1 x_2 x_3 x_1 x_3 x_2 x_3.$$

Let $s, t, u \in S^{\mathrm{I}}$.

Suppose $s = 1$. It is mere routine to check that all the substitutions

$$x_1 = 1, x_2 = t, x_3 = u$$

$$x_1 = t, x_2 = 1, x_3 = u$$

$$x_1 = t, x_2 = u, x_3 = 1$$

produce $tu = tu$. Hence $S^{\mathrm{I}} \in \mathbf{V}(R_3 = S_3)$.

We will now prove by induction that the statement is true for $\mathbf{V}(R_l = S_l), l \geq 3$. This will finish the proof of the lemma.

Let $l \geq 4$ and let $S \in \mathbf{V}(R_l = S_l)$. Then S satisfies the identity $R_l = S_l$. If l is even this means that S satisfies the identity

$$R_{l-1} x_l = S_{l-1} x_l R_{l-1} x_l.$$

Notice first that this identity implies the identity

$$R_{l-1} = S_{l-1} R_{l-1} \qquad (\text{put } x_l = R_{l-1}). \tag{3.3.2.1}$$

Now let $s_1, \ldots, s_l \in S^{\mathrm{I}}$ and consider the substitution

$$x_1 = s_1, \ldots, x_l = s_l. \tag{3.3.2.2}$$

If $s_1, \dots s_l \in S$, there is nothing to check. Suppose this does not hold.

If $s_i = 1$ for some $i \in \{1, \dots, l-1\}$, let T be the subsemigroup of S generated by $X = \{s_1, \dots, s_{l-1}\} \setminus \{1\}$. Since $|X| \leq l-2$, Corollary 3.2.6 yields that $T \in \mathbf{V}(R_{l-1} = S_{l-1})$ and the induction hypothesis gives that $T^{\mathrm{I}} \in \mathbf{V}(R_{l-1} = S_{l-1})$. Hence

$$R_{l-1}(s_1, \dots, s_{l-1}) = S_{l-1}(s_1, \dots, s_{l-1}).$$

and we get

$$R_{l-1}(s_1, \dots, s_{l-1})s_l = R_{l-1}(s_1, \dots, s_{l-1})s_l R_{l-1}(s_1, \dots, s_{l-1})s_l$$

that is,

$$R_{l-1}(s_1, \dots, s_{l-1})s_l = R_{l-1}(s_1, \dots, s_{l-1})s_l.$$

If $s_i \neq 1$ for all $i \in \{1, \dots, l-1\}$, then $s_1, \dots, s_{l-1} \in S$. Thus $s_l = 1$ and substitution (3.3.2.2) produces

$$R_{l-1}(s_1, \dots, s_{l-1}) = S_{l-1}(s_1, \dots, s_{l-1})R_{l-1}(s_1, \dots, s_{l-1})$$

which by (3.3.2.1) is a true identity.

For l odd the identity $R_l = S_l$ is equivalent to

$$x_l R_{l-1} = x_l R_{l-1} x_l S_{l-1}$$

and the proof is similar.

Remark 3.3.3. Theorem 3.3.1 does not hold for $V = \mathbf{VM}(x = 1)$ since $F_A(x = 1) = 1$ (the trivial monoid), $\chi(V) = \mathbf{V}(x = y)$, $F_A(x = y) = 1$ (the trivial semigroup) and $|\left(F_A(x = y)\right)^{\mathrm{I}}| = 2$.

Let $V \in \mathbf{LBM_0}$, $X \subseteq A$ non-empty and $\mathrm{inc} : F_A(< V >_{\mathrm{S}}) \rightarrow F_A(V)$ be the inclusion map. Since $\mathrm{inc}|_{U_X(<V>_{\mathrm{S}})} : U_X(< V >_{\mathrm{S}}) \rightarrow U_X(V)$ is an isomorphism, we identify in a natural way $U_X(V)$ with $U_X(< V >_S)$. Hence, the cardinalities we have computed, of the Green classes of the free objects in the varieties of **LB** are still the cardinalities of the Green classes of the corresponding varieties of **LBM**.

Example 3.3.4. Let $V = \mathbf{VM}(R_3 = S_3)$. Then $< V >_S = \mathbf{V}(R_3 = S_3)$ and

$$| F_A(V) | = | (F_A(< V >_S))^I | = 1 + | F_A(R_3 = S_3) | .$$

PART II – RHODES EXPANSIONS OF BANDS

CHAPTER 4. RHODES EXPANSIONS

In this chapter we use the language of the theory of categories. We deal here with semigroup expansions, a notion used in global theory of semigroups.

Section 1 is an introductory section. We present here the notion of expansion and two known examples of expansions of semigroups, namely the Rhodes expansion (the oldest expansion of semigroups known) and the Rhodes expansion cut-down to generators, a restriction of the first one, which has better properties. These expansions are defined for semigroups. It is the second one that we shall work with, in chapter 5.

In section 2 we introduce the notion of Rhodes expansion of a monoid and its cut-down to generators and we relate this notion with the corresponding one for semigroups. This will enable us to deduce (in chapter 5), for band monoids, results corresponding to the ones obtained for bands.

1. Rhodes expansions. Rhodes expansions cut-down to generators

Definition 4.1.1. An *expansion* in a category **C** is a pair (F, η), where

(1) F is a functor from **C** to **C**.

(2) η is a natural transformation from F to the functor $\mathrm{Id}_{\mathbf{C}}$, such that for every object A of **C**, $\eta_A : FA \to \mathrm{Id}_{\mathbf{C}} A$ is an epimorphism of **C**.

Definition 4.1.2. Let S be a semigroup and let

$$\overrightarrow{S}^{\mathcal{L}} = \{(x_n \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} x_1) : n \geq 1, x_1, \ldots, x_n \in S\}$$

(i.e. the set of all $\leq_{\mathcal{L}}$-chains over S).

Define in $\overline{S}^{\mathcal{L}}$ a composition $\circ$ by

$$\begin{aligned}&(x_n \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} x_1) \circ (y_m \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} y_1)\\ &=(x_n y_m \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} x_1 y_m \leq_{\mathcal{L}} y_m \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} y_1).\end{aligned}$$

Clearly $(\overline{S}^{\mathcal{L}}, \circ)$ is a semigroup.

Let

$$\hat{S}^{\mathcal{L}} = \{(x_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} x_1) : n \geq 1, x_1, \ldots, x_n \in S\}$$

(i.e., the set of all $<_{\mathcal{L}}$-chains over S). We define a mapping Red ("reduction") from $\overline{S}^{\mathcal{L}}$ to $\hat{S}^{\mathcal{L}}$ recursively, by

(1) If an $\leq_{\mathcal{L}}$-chain (or subchain) contains $<_{\mathcal{L}}$:

$$\begin{aligned}&\mathrm{Red}(x_n \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} x_{k+1} <_{\mathcal{L}} x_k \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} x_1)\\ &=(\mathrm{Red}(x_n \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} x_{k+1}) <_{\mathcal{L}} \mathrm{Red}(x_k \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} x_1)).\end{aligned}$$

(2) If an $\leq_{\mathcal{L}}$-chain (or subchain) contains only $\equiv_{\mathcal{L}}$:

$$\mathrm{Red}(x_n \equiv_{\mathcal{L}} \cdots \equiv_{\mathcal{L}} x_1) = x_n.$$

Example 4.1.3. If $a, b, c, d, e, f \in S$ then

$$\mathrm{Red}(a \equiv_{\mathcal{L}} b \equiv_{\mathcal{L}} c <_{\mathcal{L}} d <_{\mathcal{L}} e \equiv_{\mathcal{L}} f) = (a <_{\mathcal{L}} d <_{\mathcal{L}} e).$$

The mapping Red has the following crucial property:

Fact 4.1.4. $\mathrm{Red}(s \circ t) = \mathrm{Red}(\mathrm{Red}(s) \circ \mathrm{Red}(t))$, where $s, t \in \overline{S}^{\mathcal{L}}$ and $\circ$ denotes the multiplication in $\overline{S}^{\mathcal{L}}$.

We now define the Rhodes expansion of a semigroup, a notion introduced in 1969 by J. Rhodes.

Definition 4.1.5. Let S be a semigroup. The *Rhodes expansion* of S is $\hat{S}^{\mathcal{L}}$ with composition

$$st = \mathrm{Red}\,(s \circ t)$$

where $s, t \in \overline{S}^{\mathcal{L}}$ and $\circ$ denotes the multiplication in $\overline{S}^{\mathcal{L}}$.

(4.1.6.) Fact 4.1.4 shows that the composition defined in 4.1.5 is associative and the mapping Red is a morphism of semigroups.

We see that $(\hat{(.)}^{\mathcal{L}}, \eta)$, where η is defined below, is an expansion in S:

If $\varphi : S \to T$ is a morphism, define $\hat{\varphi}^{\mathcal{L}} : \hat{S}^{\mathcal{L}} \to \hat{T}^{\mathcal{L}}$ by

$$\hat{\varphi}^{\mathcal{L}}(x_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} x_1) = \mathrm{Red}\,(\varphi(x_n) \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} \varphi(x_1)).$$

It is easy to check that $\hat{\varphi}^{\mathcal{L}}$ is indeed a morphism and that $\hat{(.)}^{\mathcal{L}}$ is a functor.

If S is a semigroup, define $\eta_S : \hat{S}^{\mathcal{L}} \to S$ by

$$\eta_S(x_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} x_1) = x_n.$$

It is easy to see that η_S is a surjective morphism of S, called the *natural morphism* (according with 1.1.7).

Finally, the following diagram commutes.

$$\begin{tikzcd}
\hat{S}^{\mathcal{L}} \arrow[r, "\hat{\varphi}^{\mathcal{L}}"] \arrow[d, "\eta_S"'] & \hat{T}^{\mathcal{L}} \arrow[d, "\eta_T"] \\
S \arrow[r, "\varphi"] & T
\end{tikzcd}$$

Thus $(\hat{(.)}^{\mathcal{L}}, \eta)$ is an expansion.

We now state some properties of the Rhodes expansion, which will be used later.

Proposition 4.1.7. [1] *Let S be a semigroup. Then the natural morphism $\eta_S : \hat{S}^{\mathcal{L}} \to S$ preserves idempotents; that is, for all $s \in S$*

$$s^2 = s \text{ and } s = \eta_S(t) \Longrightarrow t^2 = t.$$

The proposition above means that the Rhodes expansion of a band is still a band. Another property of the natural morphism $\eta_S : \hat{S}^{\mathcal{L}} \to S$ is the following

Lemma 4.1.8. *Let S be a semigroup and let $x, y \in \hat{S}^{\mathcal{L}}$. Then*

$$xy = y \Longleftrightarrow \eta_S(xy) = \eta_S(y).$$

Proof. Let $x = (x_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} x_1), y = (y_m <_{\mathcal{L}} \cdots <_{\mathcal{L}} y_1), x_1, \ldots, x_n, y_1, \ldots y_m \in S$, be such that $\eta_S(xy) = \eta_S(y)$. This means that

$$\eta_S\big(\mathrm{Red}\,(x_n y_m \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} x_1 y_m \leq_{\mathcal{L}} y_m <_{\mathcal{L}} \cdots <_{\mathcal{L}} y_1)\big) = \eta_S(y_m <_{\mathcal{L}} \cdots <_{\mathcal{L}} y_1)$$

that is,

$$\eta_S\big(\mathrm{Red}\,(x_n y_m \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} x_1 y_m \leq_{\mathcal{L}} y_m <_{\mathcal{L}} \cdots <_{\mathcal{L}} y_1)\big) = y_m.$$

By the definitions of Red and η_S we get

$$\mathrm{Red}\,(x_n y_m \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} x_1 y_m \leq_{\mathcal{L}} y_m <_{\mathcal{L}} \cdots <_{\mathcal{L}} y_1) = (y_m <_{\mathcal{L}} \cdots <_{\mathcal{L}} y_1)$$

that is, $xy = y$.

The converse is trivial.

Remark 4.1.9. One can also define the Rhodes expansion with respect to $>_{\mathcal{R}}$, dually as:

$$\hat{S}^{\mathcal{R}} = \{(x_1 >_{\mathcal{R}} \cdots >_{\mathcal{R}} x_n) : n \geq 1, x_1, \ldots, x_n \in S\}$$

with multiplication:

$$\begin{aligned}(x_1 >_{\mathcal{R}} \cdots >_{\mathcal{R}} x_n) \circ (y_1 >_{\mathcal{R}} \cdots >_{\mathcal{R}} y_m) \\ = \mathrm{Red}\,(x_1 >_{\mathcal{R}} \cdots >_{\mathcal{R}} x_n \geq_{\mathcal{L}} x_n y_1 \geq_{\mathcal{R}} \cdots \geq_{\mathcal{R}} x_n y_m),\end{aligned}$$

where the mapping Red is defined recursively by

$$\mathrm{Red}\,(\ldots x_i >_{\mathcal{R}} x_{i+1} \ldots) = (\mathrm{Red}\,(\ldots x_i) >_{\mathcal{R}} \mathrm{Red}\,(x_{i+1} \ldots))$$

and

$$\mathrm{Red}\,(x_1 \equiv_{\mathcal{R}} \cdots \equiv_{\mathcal{R}} x_n \equiv_{\mathcal{R}} \cdots) = \mathrm{Red}\,(x_n \equiv_{\mathcal{R}} \cdots).$$

The properties of $(\hat{.})^{\mathcal{R}}$ are dual to those of $(\hat{.})^{\mathcal{L}}$. Moreover, if S is a semigroup, then

$$\hat{S}^{\mathcal{R}} \simeq \overline{((\hat{\overline{S}})^{\mathcal{L}})}$$

(where $\overline{S}$ is the dual of S).

Unless we specify otherwise, a Rhodes expansion will mean $(\hat{.})^{\mathcal{L}}$.

In 1.3 we have introduced the category $\mathcal{S}_A$ of semigroups generated by a given set A. Given an arbitrary expansion (F, η) in $\mathcal{S}$ one can restrict it to $\mathcal{S}_A$, obtaining an expansion (F_A, η_A), called the *cut-down* of (F, η) to the generators A. This simple idea, introduced by Rhodes [22] for the Rhodes expansion and generalized by Birget

[1] to arbitrary expansions, has important consequences, for the new expansion often has better properties. Here we present the notion in respect to the Rhodes expansion, since this is the one we shall work with, in the next chapter.

Definition 4.1.10. Let (S, f) be an object of the category S_A. The *cut-down of the Rhodes expansion of S to generators A* is the subsemigroup $\hat{S}^{\mathcal{L}}_A$ of $\hat{S}^{\mathcal{L}}$ generated by the set $\{(f(a)) : a \in A\}$.

More precisely, the image of (S, f) is the pair $(\hat{S}^{\mathcal{L}}_A, \hat{f}^{\mathcal{L}}_A)$ where $\hat{S}^{\mathcal{L}}_A$ has just been defined and $\hat{f}^{\mathcal{L}}_A : A \to \hat{S}^{\mathcal{L}}_A$ is defined by

$$\hat{f}^{\mathcal{L}}_A(a) = (f(a)) \quad (a \in A).$$

If (S, f) and (T, g) are objects of S_A and φ is a morphism from (S, f) into (T, g), we define $\hat{\varphi}^{\mathcal{L}}_A$ from $(\hat{S}^{\mathcal{L}}_A, \hat{f}^{\mathcal{L}}_A)$ into $(\hat{T}^{\mathcal{L}}_A, \hat{g}^{\mathcal{L}}_A)$ simply as

$$\hat{\varphi}^{\mathcal{L}}_A = \hat{\varphi}^{\mathcal{L}} \mid_{\hat{S}^{\mathcal{L}}_A}.$$

We obtain a natural transformation from the functor $(\hat{.})^{\mathcal{L}}_A$ to the identity functor Id_{S_A} by:

If (S, f) is an object of S_A we define $\eta_{S,A} : \hat{S}^{\mathcal{L}}_A \to S$ by

$$\eta_{S,A} = \eta_S \mid_{\hat{S}^{\mathcal{L}}_A}.$$

If $a \in A$, we have

$$\eta_{S,A}\hat{f}^{\mathcal{L}}_A(a) = \eta_{S,A}((f(a))) = f(a).$$

Thus $\eta_{S,A}$ is an epimorphism of S_A. The following diagram commutes (where the arrows denote morphisms in S_A).

$$\begin{tikzcd}
\hat{S}^{\mathcal{L}}_A \arrow[r, "\hat{\varphi}^{\mathcal{L}}_A"] \arrow[d, "\eta_{S,A}"'] & \hat{T}^{\mathcal{L}}_A \arrow[d, "\eta_{T,A}"] \\
S \arrow[r, "\varphi"] & T
\end{tikzcd}$$

Hence $((\hat{\cdot})^{\mathcal{L}}_A, \eta_A)$ is an expansion in $\mathcal{S}_A$.

Fact 4.1.11. If S is a band, then $\hat{S}^{\mathcal{L}}_A$ is also a band. (See 4.1.7.)

Remark 4.1.12. We can define $(\hat{\cdot})^{\mathcal{R}}_A$, the *cut-down to generators* A of $(\hat{\cdot})^{\mathcal{R}}$, and obtain an expansion with properties dual to those of $(\hat{\cdot})^{\mathcal{L}}$.

Remark 4.1.13. If $V \in \mathbf{LB_0}$ and $S = F_A(V)$, $S \simeq t_k(A^+)$, $t \in \{h, i\}$ (see 2.2.13), we sometimes denote $[u]_{\nu(V)} \in F_A(V)$ by $t_k(u)$ or just by u, when clear.

Example 4.1.14. Let $A = \{a, b\}$ and $S = F_A(xy = yx)$. If $u, v \in A^+$, by 1.5.1 and 2.2.10 we have

$$\begin{aligned} u \leq_{\mathcal{L}} v &\iff c(uv) = c(u) \\ &\iff c(u) \supseteq c(v) \end{aligned}$$

and

$$u <_{\mathcal{L}} v \iff c(u) \supset c(v)$$

where the last inclusion is strict.

Hence

$$\hat{S}^{\mathcal{L}} = \{(a), (b), (ab), (ab <_{\mathcal{L}} a), (ab <_{\mathcal{L}} b)\}.$$

Now,

$$\hat{S}^{\mathcal{L}}_A = <(a), (b)> .$$

Since

$$\begin{aligned} &(b)(a) = (ab <_{\mathcal{L}} a) \\ &(a)(b) = (ab <_{\mathcal{L}} b) \\ &(a)(b)(a) = \mathrm{Red}(aba \leq_{\mathcal{L}} ab <_{\mathcal{L}} a) = (ab <_{\mathcal{L}} a) \end{aligned}$$

$$(b)(a)(b) = \mathrm{Red}(bab \leq_{\mathcal{L}} ab <_{\mathcal{L}} b) = (ab <_{\mathcal{L}} b)$$

we get

$$\hat{S}_A^{\mathcal{L}} = \{(a), (b), (ab <_{\mathcal{L}} a), (ab <_{\mathcal{L}} b)\}.$$

We finish this section with a known result concerning the effect of applying $(\hat{.})_A^{\mathcal{L}}$ to a semigroup a finite number of times.

Proposition 4.1.15. [Tilson, 27] *Let S be a semigroup. Then*

$$\left(\hat{\hat{S}}_A^{\mathcal{L}}\right)_A^{\mathcal{L}} \simeq \hat{S}_A^{\mathcal{L}} \quad \left(\textit{and dually } \left(\hat{\hat{S}}_A^{\mathcal{R}}\right)_A^{\mathcal{R}} \simeq \hat{S}_A^{\mathcal{R}}\right).$$

A more detailed treatment of the notions presented in this section can be found in [1], [2] and [27].

2. Rhodes expansions of monoids

Here we establish the notion of Rhodes expansion of a monoid and its cut-down to generators. We relate these notions with the ones presented in the previous section for semigroups.

Definition 4.2.1. Let M be a monoid. The *Rhodes expansion of M* is the set

$$\hat{M}_e^{\mathcal{L}} = \{(x_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} x_1) : n \geq 1, x_1, \ldots, x_n \in M, x_1 \mathcal{L} \mathrm{id}_M\}.$$

Clearly, $\hat{M}_e^{\mathcal{L}}$ is a subsemigroup of $\hat{M}^{\mathcal{L}}$ and it is a monoid.

If M and N are monoids and $\varphi : M \to N$ is a monoid morphism then $\hat{\varphi}^{\mathcal{L}}(\hat{M}_e^{\mathcal{L}}) \subseteq \hat{N}_e^{\mathcal{L}}$. We define $\hat{\varphi}_e^{\mathcal{L}} : \hat{M}_e^{\mathcal{L}} \to \hat{N}_e^{\mathcal{L}}$ by

$$\hat{\varphi}_e^{\mathcal{L}} = \hat{\varphi}^{\mathcal{L}} |_{\hat{M}_e^{\mathcal{L}}}$$

(where $\hat{\varphi}^{\mathcal{L}} : \hat{M}^{\mathcal{L}} \to \hat{N}^{\mathcal{L}}$ is defined in $\mathcal{S}$).

We get a natural transformation from the functor $(\hat{.})_e^{\mathcal{L}}$ of $\mathcal{M}$ to the functor $\mathrm{Id}_{\mathcal{M}}$ in the following way:

If M is a monoid, we define $\eta_{M,e} : \hat{M}_e^{\mathcal{L}} \to M$ by

$$\eta_{M,e} = \eta_M \mid_{\hat{M}_e^{\mathcal{L}}}$$

(where $\eta_M : \hat{M}^{\mathcal{L}} \to M$ is defined in $\mathcal{S}$).

Clearly, $\eta_{M,e}$ is an epimorphism of $\mathcal{M}$ and the following diagram commutes:

$$\begin{tikzcd}
\hat{M}_e^{\mathcal{L}} \arrow[r, "\hat{\varphi}_e^{\mathcal{L}}"] \arrow[d, "\eta_{M,e}"'] & \hat{N}_e^{\mathcal{L}} \arrow[d, "\eta_{N,e}"] \\
M \arrow[r, "\varphi"] & N
\end{tikzcd} \tag{4.2.1.1}$$

Hence $((\hat{.})_e^{\mathcal{L}}, \eta_e)$ is an expansion in $\mathcal{M}$.

We define the cut-down to generators of this expansion in the following way:

Definition 4.2.2. Let (M, f) be an object of $\mathcal{M}_A$. The *cut-down of the Rhodes expansion of M to generators A* is

$$\hat{M}_{e,A}^{\mathcal{L}} =< \mathrm{Red}\,(f(a) \leq_{\mathcal{L}} \mathrm{id}_M) : a \in A >_{\hat{M}_e^{\mathcal{L}}}$$

(i.e., the submonoid of $\hat{M}_e^{\mathcal{L}}$ generated by the set $\{(f(a) \leq_{\mathcal{L}} \mathrm{id}_M) : a \in A\}$).

More precisely, the image of (M, f) is the pair $(\hat{M}_{e,A}^{\mathcal{L}}, \hat{f}_{e,A}^{\mathcal{L}})$ where $\hat{M}_{e,A}^{\mathcal{L}}$ was defined above and $\hat{f}_{e,A}^{\mathcal{L}} : A \to \hat{M}_{e,A}^{\mathcal{L}}$ is defined by

$$\hat{f}_{e,A}^{\mathcal{L}}(a) = \mathrm{Red}\,(f(a) \leq_{\mathcal{L}} \mathrm{id}_M) \quad (a \in A).$$

If (M, f) and (N, g) are objects of $\mathcal{M}_A$ and φ is a morphism from (M, f) into (N, g), then $\hat{\varphi}_e^{\mathcal{L}}(\hat{M}_{e,A}^{\mathcal{L}}) \subseteq \hat{N}_{e,A}^{\mathcal{L}}$ and we define $\hat{\varphi}_{e,A}^{\mathcal{L}}$ from $\hat{M}_{e,A}^{\mathcal{L}}$ into $\hat{N}_{e,A}^{\mathcal{L}}$ by

$$\hat{\varphi}_{e,A}^{\mathcal{L}} = \hat{\varphi}_e^{\mathcal{L}} \mid_{\hat{M}_{e,A}^{\mathcal{L}}} .$$

We obtain a natural transformation from $\hat{(.)}_{e,A}^{\mathcal{L}}$ to the functor $\mathrm{Id}_{\mathcal{M}_A}$ in the following way:

If M is a monoid we define $\eta_{M,e,A} : \hat{M}_{e,A}^{\mathcal{L}} \to M$ by

$$\eta_{M,e,A} = \eta_{M,e} \mid_{\hat{M}_{e,A}^{\mathcal{L}}} .$$

If $a \in A$, we have

$$\eta_{M,e,A} \hat{f}_{e,A}^{\mathcal{L}}(a) = \eta_{M,e,A}(\mathrm{Red}(f(a) \leq_{\mathcal{L}} \mathrm{id}_M)) = f(a).$$

Hence $\eta_{M,e,A}$ is an epimorphism of $\mathcal{M}_A$. The following diagram is commutative (where the arrows denote morphisms of $\mathcal{M}_A$).

$$\begin{tikzcd} \hat{M}_{e,A}^{\mathcal{L}} \arrow[r, "\hat{\varphi}_{e,A}^{\mathcal{L}}"] \arrow[d, "\eta_{M,e,A}"'] & \hat{N}_{e,A}^{\mathcal{L}} \arrow[d, "\eta_{N,e,A}"] \\ M \arrow[r, "\varphi"] & N \end{tikzcd}$$

Hence, $\left(\hat{(.)}_{e,A}^{\mathcal{L}}, \eta_{e,A}\right)$ is an expansion in $\mathcal{M}_A$.

Example 4.2.3. Let M be the free object in $\mathbf{VM}(xy = yx)$ generated by $A = \{a, b\}$. Then

$$\hat{M}_e^{\mathcal{L}} = \{(a <_{\mathcal{L}} 1), (b <_{\mathcal{L}} 1), (ab <_{\mathcal{L}} 1), (ab <_{\mathcal{L}} a <_{\mathcal{L}} 1), (ab <_{\mathcal{L}} b <_{\mathcal{L}} 1)\}$$

and

$$\hat{M}_{e,A}^{\mathcal{L}} = \{(a <_{\mathcal{L}} 1), (b <_{\mathcal{L}} 1), (ab <_{\mathcal{L}} a <_{\mathcal{L}} 1), (ab <_{\mathcal{L}} b <_{\mathcal{L}} 1)\}.$$

Notice that we are identifying again $c(u)$ with u.

We shall see in the next chapter that the results concerning the Rhodes expansions of band monoids are easily deducible from the ones concerning the Rhodes expansions of bands. For that purpose we need to explore the relationships between the two expansions. This we do next.

Remark 4.2.4. Let (S, f) be a semigroup and let $S^{\mathrm{I}} = S \cup \{1_S\}$ (as defined in 1.2.1). For all $s \in S$, $s <_{\mathcal{L}} 1_S$. Otherwise, since $s \leq_{\mathcal{L}} 1_S$, we would get sucessively $s \equiv_{\mathcal{L}} 1_S$, $1_S = ts$ for some $t \in S^{\mathrm{I}}$, $1_S \in S$.

Theorem 4.2.5. *The functors* $(.)^{\mathrm{I}} \circ \hat{(.)}_A^{\mathcal{L}}$ *and* $\hat{(.)}_{e,A}^{\mathcal{L}} \circ (.)^{\mathrm{I}}$ *are naturally equivalent; that is,*

$$(.)^{\mathrm{I}} \circ \hat{(.)}_A^{\mathcal{L}} \approx \hat{(.)}_{e,A}^{\mathcal{L}} \circ (.)^{\mathrm{I}}.$$

In particular, for any semigroup S we have

$$(\hat{S}^{\mathrm{I}})_{e,A}^{\mathcal{L}} \simeq (\hat{S}_A^{\mathcal{L}})^{\mathrm{I}}.$$

Proof. Let (S, f) and (T, g) be semigroups and let $\varphi : (S, f) \to (T, g)$ be a morphism. We define $\xi_S : \hat{S}^{\mathcal{L}} \to (\hat{S}^{\mathrm{I}})_e^{\mathcal{L}}$ by

$$\xi_S\big((s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1)\big) = (s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1 <_{\mathcal{L}} 1_S).$$

Clearly, $\xi_S(\hat{S}_A^{\mathcal{L}}) \subseteq (\hat{S}^{\mathrm{I}})_{e,A}^{\mathcal{L}}$, since the image of a generator $(f(a))$ $(a \in A)$ of $\hat{S}_A^{\mathcal{L}}$ is the generator $(f(a) <_{\mathcal{L}} 1_S)$ of $(\hat{S}^{\mathrm{I}})_{e,A}^{\mathcal{L}}$. Hence, we define $\xi_{S,A} : \hat{S}_A^{\mathcal{L}} \to (\hat{S}^{\mathrm{I}})_{e,A}^{\mathcal{L}}$ by

$$\xi_{S,A} = \xi_S \mid_{\hat{S}_A^{\mathcal{L}}}.$$

Clearly, $\xi_{S,A}$ is injective and $\mathrm{Im}(\xi_{S,A}) = (\hat{S}^{\mathrm{I}})_{e,A}^{\mathcal{L}} \setminus \{(1_S)\}$.

Hence $\xi_{S,e,A} : (\hat{S}_A^{\mathcal{L}})^{\mathrm{I}} \to (\hat{S}^{\mathrm{I}})_{e,A}^{\mathcal{L}}$ defined by

$$\xi_{S,e,A}(s) = \begin{cases} \xi_{S,A}(s), & \text{if } s \in \hat{S}_A^{\mathcal{L}}; \\ (1_S), & \text{if } s = 1_{\hat{S}_A^{\mathcal{L}}}; \end{cases}$$

is an isomorphism.

Now let $\hat{\varphi}_A^{\mathcal{L}} = \hat{\varphi}^{\mathcal{L}} \mid_{\hat{S}_A^{\mathcal{L}}}$ and $(\hat{\varphi}^{\mathrm{I}})_{e,A}^{\mathcal{L}} = (\hat{\varphi}^{\mathrm{I}})^{\mathcal{L}} \mid_{(\hat{S}^{\mathrm{I}})_{e,A}^{\mathcal{L}}}$. Since $\varphi^{\mathcal{L}}(\hat{S}_A^{\mathcal{L}}) \subseteq \hat{T}_A^{\mathcal{L}}$ and $(\hat{\varphi}^{\mathrm{I}})^{\mathcal{L}}\big((\hat{S}^{\mathrm{I}})_{e,A}^{\mathcal{L}}\big) \subseteq (\hat{T}^{\mathrm{I}})_{e,A}^{\mathcal{L}}$ the following diagram is commutative:

$$\begin{tikzcd} (\hat{S}_A^{\mathcal{L}})^{\mathrm{I}} \arrow[r, "(\hat{\varphi}_A^{\mathcal{L}})^{\mathrm{I}}"] \arrow[d, "\xi_{S,e,A}"'] & (\hat{T}_A^{\mathcal{L}})^{\mathrm{I}} \arrow[d, "\xi_{T,e,A}"] \\ (\hat{S}^{\mathrm{I}})_{e,A}^{\mathcal{L}} \arrow[r, "(\hat{\varphi}^{\mathrm{I}})_{e,A}^{\mathcal{L}}"] & (\hat{T}^{\mathrm{I}})_{e,A}^{\mathcal{L}} \end{tikzcd}$$

Since $\xi_{S,e,A}$ and $\xi_{T,e,A}$ are isomorphisms, we deduce in particular that $(\hat{S}^{\mathrm{I}})_{e,A}^{\mathcal{L}} \simeq (\hat{S}_A^{\mathcal{L}})^{\mathrm{I}}$.

CHAPTER 5. RHODES EXPANSIONS OF BANDS

In this chapter we consider Birkhoff varieties of bands.

In the first section we compute the cut-down to generators of the Rhodes expansions of the free objects in the varieties of bands.

In section 2 it is shown how to deduce, for band monoids, results corresponding to the ones obtained in the first section.

1. Rhodes expansions of the free objects in the varieties of bands

As we have done in the third chapter, we study first the varieties of rectangular bands.

Proposition 5.1.1. *If S is a rectangular band, then $\hat{S}_A^{\mathcal{L}} \simeq S$.*

Proof. The statement is clear, since by 1.5.6 we deduce that all the sequences $(s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1)$ of $\hat{S}_A^{\mathcal{L}}$ have length one.

Thus if V is one of the varieties

$$\mathbf{V}(x=y), \mathbf{V}(ax=a), \mathbf{V}(xa=a), \mathbf{V}(axa=a),$$

then $(F_A(V))_A^{\mathcal{L}} \simeq F_A(V)$.

From now on we work on $\mathbf{LB_0}$.

Remark 5.1.2. If $V \in \mathbf{LB_0}$ is a proper variety, then $F_A(V) \simeq t_k(A^+)$. If $u, v \in A^+$

are such that $c(u) = c(v)$, then by 2.2.5 and 2.2.16, $t_k(u)\mathcal{J}t_k(v)$ and 1.5.7 yields

$$t_k(u) \leq_{\mathcal{L}} t_k(v) \Rightarrow t_k(u)\mathcal{L}t_k(v).$$

Example 5.1.3. Let $A = \{a, b\}$ and let $S = F_A(xy = yx)$. We have seen in 4.1.14 that

$$\hat{S}_A^{\mathcal{L}} = \{(a), (b), (ab <_{\mathcal{L}} a), (ab <_{\mathcal{L}} b)\}.$$

It is mere routine to check that $\hat{S}_A^{\mathcal{L}} \in \mathbf{V}(axa = xa)$. Hence there is a unique (surjective) morphism $\varphi : F_A(axa = xa) \to \hat{S}_A^{\mathcal{L}}$ such that the following diagram is commutative. (See 2.1.17.)

$$\begin{tikzcd}
A \arrow[r, "\natural"] \arrow[d, "\hat{\natural}_A^{\mathcal{L}}"'] & F_A(axa = xa) \arrow[dl, "\varphi"] \\
\hat{S}_A^{\mathcal{L}} &
\end{tikzcd}$$

On the other hand, $\mid F_A(axa = xa) \mid = 4$. Therefore $\hat{S}_A^{\mathcal{L}} \simeq F_A(axa = xa)$. This remains true for any $\mid A \mid$ finite. In order to show this, we will show that:

(1) $\hat{S}^{\mathcal{L}} \in \mathbf{V}(axa = xa)$ (hence $\hat{S}_A^{\mathcal{L}} \in \mathbf{V}(axa = xa)$).

(2) $\hat{S}_A^{\mathcal{L}} = \{(s_k <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) : k \geq 1,\ x_i \in A^+,\ \mid c(s_i) \mid = i,\ i = 1, \ldots, k\}$
(where $(s_k <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1)$ denotes $([s_k]_{\nu(V)} <_{\mathcal{L}} \cdots <_{\mathcal{L}} [s_1]_{\nu(V)})$).

(3) $\mid \hat{S}_A^{\mathcal{L}} \mid = \mid F_A(axa = xa) \mid$.

Proof of (1). Let $s, t \in \hat{S}^{\mathcal{L}}$, $s = (s_k <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1)$, $t = (t_l <_{\mathcal{L}} \cdots <_{\mathcal{L}} t_1)$, where $k, l \geq 1$, $s_i, t_j \in A^+$, $\mid c(s_i) \mid = i$, $\mid c(t_j) \mid = j$, $1 \leq i \leq k$, $1 \leq j \leq l$. Then

$$\begin{aligned}
sts &= \mathrm{Red}(s_k t_l \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} s_1 t_l <_{\mathcal{L}} t_l <_{\mathcal{L}} \cdots <_{\mathcal{L}} t_1) \,.\, (s_k <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) \\
&= \mathrm{Red}(s_k t_l s_k \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} s_1 t_l s_k \leq_{\mathcal{L}} t_l s_k \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} t_1 s_k \leq_{\mathcal{L}} s_k <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) \\
&= \mathrm{Red}(t_l s_k \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} t_1 s_k \leq_{\mathcal{L}} s_k <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) \\
&= ts
\end{aligned}$$

since $s_k t_l s_k \mathcal{L} t_l s_k$ (because $c(s_k t_l s_k) = c(t_l s_k)$).

Proof of (2). Let $X = \{(s_k <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) : k \geq 1,\ s_i \in A^+,\ \mid c(s_i) \mid = i,\ i = 1, \ldots, k\}$.

Let $s \in \hat{S}_A^{\mathcal{L}}$. If $s = (a),\ a \in A$, then $s \in X$. Suppose that any product $(a_n) \ldots (a_1),\ n \geq 1,\ a_i \in A,\ i = 1, \ldots, n$ is in X. Let $a_{n+1} \in A$. Then

$$\begin{aligned}(a_{n+1}) \ldots (a_1) &= (a_{n+1})[(a_n) \ldots (a_1)] \\ &= (a_{n+1})(s_k <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) \qquad (5.1.3.1)\end{aligned}$$

by the induction hypothesis, where $s_i \in A^+,\ \mid c(s_i) \mid = i,\ i = 1, \ldots, k$. Hence

$$(5.1.3.1) = \mathrm{Red}\,(a_{n+1}\, s_k \leq_{\mathcal{L}} s_k <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1).$$

If $a_{n+1} \notin c(s_k)$, then $a_{n+1}\, s_k <_{\mathcal{L}} s_k$ and

$$(5.1.3.1) = (a_{n+1}\, s_k <_{\mathcal{L}} s_k <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1),$$

with $\mid c(a_{n+1}\, s_k) \mid = k + 1$.

If $a_{n+1} \in c(s_k)$, then $a_{n+1}\, s_k \mathcal{L} s_k$ and

$$(5.1.3.1) = (a_{n+1}\, s_k <_{\mathcal{L}} s_{k-1} <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1),$$

where $\mid c(a_{n+1}\, s_k) \mid = k$.

Thus $\hat{S}_A^{\mathcal{L}} \subseteq X$.

Conversely, let $s = (s_k <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) \in X$ and let $\{a_i\} = c(s_i) \setminus c(s_{i-1}),\ i = 1, \ldots, k$. In S, $s_i = a_i s_{i-1},\ i = 1, \ldots, k$ (since $c(s_i) = c(a_i s_{i-1})$). It is then a mere routine to check that

$$s = (a_k) \ldots (a_1).$$

Thus, $s \in \hat{S}_A^{\mathcal{L}}$.

Proof of (3). Let $\mid A \mid = N$. By (2), we have

$$
\begin{aligned}
\mid \hat{S}_A^{\mathcal{L}} \mid &= \mid \{(s_k <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) : k \geq 1, s_i \in A^+, \mid c(s_i) \mid = i, i = 1, \ldots, k\} \mid \\
&= \sum_{k=1}^{N} \mid \{(s_k, \ldots, s_1) : s_i \in A^+, c(s_i) \supseteq c(s_{i-1}), \mid c(s_i) \mid = i, i = 1, \ldots, k\} \mid \\
&= \sum_{k=1}^{N} N(N-1) \ldots (N-k+1) \\
&= \sum_{k=1}^{N} \binom{N}{k} k! \\
&= \mid F_A(axa = xa) \mid . \quad \text{(See 3.1.1 and 3.1.10.)}
\end{aligned}
$$

From (1)–(3) and 2.1.17, we deduce that $\hat{S}_A^{\mathcal{L}} \simeq F_A(axa = xa)$.

We now state the main theorem of this chapter.

Theorem 5.1.4. *If* $V \in \mathbf{LB_0}$ *and* $S = F_A(V)$ *then*

$$\hat{S}_A^{\mathcal{L}} \simeq F_A(V^r).$$

For instance, if $S = F_A(V)$ where

$$V \in \{\mathbf{V}(axa = xa), \mathbf{V}(\overline{R}_2 = \overline{S}_2), \mathbf{V}(axya = axaya), \mathbf{V}(\overline{R}_2 dR_3 = S_2 d\overline{Q}_3)\},$$

Theorem 5.1.4 asserts that $\hat{S}_A^{\mathcal{L}} = F_A(\overline{R}_3 = \overline{S}_3)$.

We now give a statement that will reduce the number of cases we have to consider when proving Theorem 5.1.4. Indeed, with this result it is enough to prove the theorem for the left varieties of $\mathbf{LB_0}$.

Proposition 5.1.5. *Let $S, T, U \in \mathcal{S}_A$ and let $\varphi : S \to T$, $\psi : T \to U$ be morphisms. Then*

$$\hat{U}_A^{\mathcal{L}} \simeq S \Rightarrow \hat{T}_A^{\mathcal{L}} \simeq S.$$

Proof. Let $S, T, U \in \mathcal{S}_A$ and let $\varphi : S \to T, \psi : T \to U$ be morphisms. Suppose that $\hat{U}_A^{\mathcal{L}} \simeq S$ and let χ be an isomorphism from $\hat{U}_A^{\mathcal{L}}$ into S.

By 4.1.15, $\hat{S}_A^{\mathcal{L}} \simeq S$ and by 1.3.2 the natural morphism $\eta_{S,A} : \hat{S}_A^{\mathcal{L}} \to S$ is an isomorphism. Let $\eta_{S,A}^{-1}$ be its inverse. Let $\hat{\varphi}_A^{\mathcal{L}} : \hat{S}_A^{\mathcal{L}} \to \hat{T}_A^{\mathcal{L}}$ and $\hat{\psi}_A^{\mathcal{L}} : \hat{T}_A^{\mathcal{L}} \to \hat{U}_A^{\mathcal{L}}$ be the morphisms determined by φ and ψ, respectively. (See 4.1.6 and 4.1.10.) Then

$$\hat{\varphi}_A^{\mathcal{L}} \circ \eta_{S,A}^{-1} \circ \chi \circ \hat{\psi}_A^{\mathcal{L}}$$

is a morphism from $\hat{T}_A^{\mathcal{L}}$ into itself. Therefore 1.3.2 yields again that

$$\hat{\varphi}_A^{\mathcal{L}} \circ \eta_{S,A}^{-1} \circ \chi \circ \hat{\psi}_A^{\mathcal{L}} = \mathrm{id}_{\hat{T}_A^{\mathcal{L}}}.$$

Hence $\chi \circ \hat{\psi}_A^{\mathcal{L}}$ is an isomorphism from $\hat{T}_A^{\mathcal{L}}$ into S. Thus, $\hat{T}_A^{\mathcal{L}} \simeq S$.

We now state a series of lemmas, whose proofs we shall defer until we have shown how they lead to a proof of Theorem 5.1.4.

Lemma 5.1.6. *If V is a left variety and $S \in V$, then $\hat{S}^{\mathcal{L}} \in V^{\tau}$.*

This lemma asserts that for instance if $S \in V = \mathbf{V}(R_3 = S_3)$, then $\hat{S}^{\mathcal{L}} \in \mathbf{V}(R_4 = S_4)$ (thus $\hat{S}_A^{\mathcal{L}} \in \mathbf{V}(R_4 = S_4)$).

We aim now to describe the elements of $\hat{S}_A^{\mathcal{L}}$ (when $S = F_A(V)$ and V is a proper left variety). In order to do this, we study the relation $\leq_{\mathcal{L}}$ in $F_A(V)$.

Lemma 5.1.7. *Let $u, v \in A^+$ be such that $c(u) \supseteq c(v)$, $|\, c(u) \,| = |\, c(v) \,| + 1$.*

(i) *If* V *is one of the varieties* $\mathbf{V}(\overline{R}_2 = \overline{Q}_2), \mathbf{V}(ax = axa)$, *then* $u <_{\mathcal{L}} v$ *in* $F_A(V)$.

(ii) *If* $T \in \{Q, S\},\ k \geq 3,\ V = \mathbf{V}(R_k = T_k)$ *for k odd and* $V = \mathbf{V}(\overline{R}_k = \overline{T}_k)$ *for k even, then*

$$t_k(u) \leq_{\mathcal{L}} t_k(v) \Longleftrightarrow \overline{t}_{k-1}(e(u))\mathcal{L}\overline{t}_{k-1}(v),$$

where $t = i$ *if* $T = S$ *and* $t = h$ *if* $T = Q$.

In particular, if $k = 3$

$$i_3(u) \leq_{\mathcal{L}} i_3(v) \Longleftrightarrow \overline{i}_2(e(u)) = \overline{i}_2(v)$$

and

$$h_3(u) \leq_{\mathcal{L}} h_3(v) \Longleftrightarrow \overline{h}_2(u) = \overline{h}_2(v).$$

We present next a description of $\hat{S}_A^{\mathcal{L}}$.

Proposition 5.1.8. (i) *If* V *is one of the varieties* $\mathbf{V}(\overline{R}_2 = \overline{Q}_2), \mathbf{V}(ax = axa)$, *then*

$$\hat{S}_A^{\mathcal{L}} = \{(s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) : n \geq 1, s_i \in A^+, |c(s_i)| = i, i = 1, \ldots, n\}.$$

(ii) *If* $T \in \{Q, S\}, k \geq 3, V = \mathbf{V}(R_k = T_k)$ *for k odd and* $V = \mathbf{V}(\overline{R}_k = \overline{T}_k)$ *for k even, then*

$$\hat{S}_A^{\mathcal{L}} = \{(s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) : n \geq 1, s_i \in A^+, |c(s_i)| = i, \overline{t}_{k-1}(e(s_i)) = \overline{t}_{k-1}(s_{i-1}), \quad i = 1, \ldots, n\},$$

where $t = i$ *if* $T = S$ *and* $t = h$ *if* $T = Q$.

Notice that we are using the notation introduced in 4.1.13. For instance, if $V = \mathbf{V}(R_3 = S_3)$, $(s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1)$ denotes $(i_3(s_n) <_{\mathcal{L}} \cdots <_{\mathcal{L}} i_3(s_1))$ or $([s_n]_{\nu(V)} <_{\mathcal{L}} \cdots <_{\mathcal{L}} [s_1]_{\nu(V)})$.

The next two lemmas are partial results, useful for proving the existence of an isomorphism between $\hat{S}_A^{\mathcal{L}}$ and $F_A(V^\tau)$.

Lemma 5.1.9. *Let $n \geq 1$ and let $u \in A^+$ be such that $|\, c(u) \,| = n$.*

(i) *Let*

$$Y = \{(u \leq_{\mathcal{L}} s_{n-1} \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} s_1) : s_i \in A^+, |\, c(s_i) \,| = i, i = 1, \ldots, n-1\}.$$

If $V = \mathbf{V}(\overline{R}_2 = \overline{Q}_2)$, then $|Y| = \dfrac{c_n(\overline{R}_3 = \overline{Q}_3)}{c_n(\overline{R}_2 = \overline{Q}_2)}$; if $V = \mathbf{V}(ax = axa)$, then $|Y| = \dfrac{c_n(\overline{R}_3 = \overline{S}_3)}{c_n(ax = axa)}$.

(ii) *If $T \in \{Q, S\}, k \geq 3, V = \mathbf{V}(R_k = T_k)$ for k odd and $V = \mathbf{V}(\overline{R}_k = \overline{T}_k)$ for k even, then the subset of $\hat{S}^{\mathcal{L}}$*

$$\{(u \leq_{\mathcal{L}} s_{n-1} \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} s_1) : s_i \in A^+, |c(s_i)| = i, \bar{t}_{k-1}(e(u)) = \bar{t}_{k-1}(s_{n-1}),$$
$$\bar{t}_{k-1}(e(s_i)) = \bar{t}_{k-1}(s_{i-1}), i = 1, \ldots, n-1\},$$

where $t = i$ if $T = S$ and $t = h$ if $T = Q$, has cardinality

$$\frac{c_n(R_{k+1} = T_{k+1})}{c_n(R_k = T_k)} \quad \left(= \frac{c_n(\overline{R}_{k+1} = \overline{T}_{k+1})}{c_n(\overline{R}_k = \overline{T}_k)}\right).$$

Lemma 5.1.10. *Let $n \geq 1$ and let $A_n \subseteq A$ be such that $|\, A_n \,| = n$. Let*

$$X = \{s \in \hat{S}_A^{\mathcal{L}} : c(\eta_{S,A}(s)) = A_n\}.$$

(i) *If $V = \mathbf{V}(\overline{R}_2 = \overline{Q}_2)$, then $|X| = c_n(\overline{R}_3 = \overline{Q}_3)$; if $V = \mathbf{V}(ax = axa)$, then $|X| = c_n(\overline{R}_3 = \overline{S}_3)$.*

(ii) *If $T \in \{Q, S\}, k \geq 3, V = \mathbf{V}(R_k = T_k)$ for k odd and $V = \mathbf{V}(\overline{R}_k = \overline{T}_k)$ for k even, where $t = i$ if $T = S$ and $t = h$ if $T = Q$, then X has cardinality $c_n(R_{k+1} = T_{k+1})$ $\left(= c_n(\overline{R}_{k+1} = \overline{T}_{k+1})\right)$.*

Notice that if A is finite, 5.1.10 yields that $\hat{S}_A^{\mathcal{L}}$ and $F_A(V^{\tau})$ have the same cardinality, thus they are isomorphic.

We show now how these results are used to prove Theorem 5.1.4.

Proof of Theorem 5.1.4. We first list the different cases to be considered:

(1) $V = \mathbf{B}$. In this case $\hat{S}_A^{\mathcal{L}} \simeq S$;

(2) $V = \mathbf{V}(R_k = T_k)$, k odd, $k \geq 3, T \in \{Q, S\}$. In this case, $\hat{S}_A^{\mathcal{L}} \simeq F_A(R_{k+1} = T_{k+1})$;

(3) $V = \mathbf{V}(\overline{R}_k = \overline{T}_k)$, k even, $T \in \{Q, S\}$, $k \geq 2$ if $T = Q$, $k \geq 4$ if $T = S$. In this case $\hat{S}_A^{\mathcal{L}} \simeq F_A(\overline{R}_{k+1} = \overline{T}_{k+1})$.

(4) $V = \mathbf{V}(xy = yx)$. In this case $\hat{S}_A^{\mathcal{L}} \simeq F_A(xa = axa)$;

(5) $V = \mathbf{V}(ax = axa)$. In this case, $\hat{S}_A^{\mathcal{L}} \simeq F_A(\overline{R}_3 = \overline{S}_3)$.

We now prove (1)–(5).

(1) By 4.1.11, $\hat{S}_A^{\mathcal{L}} \in \mathbf{B}$. Consequently, there is a morphism $\psi : S \to \hat{S}_A^{\mathcal{L}}$. Since $\eta_{S,A}$ is a morphism from $\hat{S}_A^{\mathcal{L}}$ into S, 1.3.4 yields that $\varphi, \eta_{S,A}$ are isomorphisms and so $\hat{S}_A^{\mathcal{L}} \simeq S$.

(2) Let k be odd, $k \geq 3$ and let $S = F_A(R_k = S_k)$. By 5.1.6 there is a surjective morphism $\varphi : F_A(R_{k+1} = S_{k+1}) \to \hat{S}_A^{\mathcal{L}}$ such that the following diagram is commutative.

$$\begin{tikzcd} A^+ \arrow[r, "\natural_{k+1}"] \arrow[d, "(\hat{\natural}_k)_A^{\mathcal{L}}"'] & F_A(R_{k+1} = S_{k+1}) \arrow[dl, "\varphi"] \\ \hat{S}_A^{\mathcal{L}} & \end{tikzcd} \qquad (5.1.4.1)$$

where $\natural_k$ denotes the canonical epimorphism from A^+ into $F_A(R_k = S_k)$. Notice that

$(\hat{\natural}_k)_A^{\mathcal{L}}$ is defined by $(\hat{\natural}_k)_A^{\mathcal{L}}(a) = (\natural_k(a)) = [a]_{\nu(R_k=S_k)}$ $(a \in A)$. (See 4.1.10.)

Also the following diagram is commutative. (See 4.1.10 again.)

$$\begin{tikzcd}
A^+ \arrow[r, "\natural_k"] \arrow[d, "(\hat{\natural}_k)_A^{\mathcal{L}}"'] & S = F_A(R_k = S_k) \\
\hat{S}_A^{\mathcal{L}} \arrow[ur, "\eta_{S,A}"'] &
\end{tikzcd} \tag{5.1.4.2}$$

We now show that φ is injective. It is enough to show that the restriction of φ to the $\mathcal{J}$-classes of $F_A(R_{k+1} = S_{k+1})$ is injective. Indeed, if t and $t' \in F_A(R_{k+1} = S_{k+1})$ are such that $\varphi(t) = \varphi(t')$, then there are $u, v \in A^+$ such that $t = \natural_{k+1}(u)$, $t' = \natural_{k+1}(v)$ and so we get

$$\begin{aligned}
\varphi(t) = \varphi(t') &\Rightarrow \varphi\natural_{k+1}(u) = \varphi\natural_{k+1}(v) \\
&\Rightarrow (\hat{\natural}_k)_A^{\mathcal{L}}(u) = (\hat{\natural}_k)_A^{\mathcal{L}}(v) \quad \text{by (5.1.4.1)} \\
&\Rightarrow \eta_{S,A}(\hat{\natural}_k)_A^{\mathcal{L}}(u) = \eta_{S,A}(\hat{\natural}_k)_A^{\mathcal{L}}(v) \\
&\Rightarrow \natural_k(u) = \natural_k(v) \quad \text{by (5.1.4.2)} \\
&\Rightarrow c(u) = c(v)
\end{aligned}$$

and therefore t and t' are $\mathcal{J}$-related in $F_A(R_{k+1} = S_{k+1})$.

Let $A_n \subseteq A$. We have

$$\begin{aligned}
&\varphi(\{t \in F_A(R_{k+1} = S_{k+1}) : c(t) = A_n\}) \\
&\quad = \{\varphi(t) \in \hat{S}_A^{\mathcal{L}} : c(t) = A_n\}
\end{aligned} \tag{5.1.4.3}$$

But if $t = \natural_{k+1}(u)$, $u \in A^+$ we get

$$c(t) = c(\natural_{k+1}(u))$$

$$
\begin{aligned}
&= c(\natural_k(u)) \\
&= c(\eta_{S,A}(\hat{\natural}_k)_A^{\mathcal{L}}(u)) \quad \text{by (5.1.4.2)} \\
&= c\big(\eta_{S,A}\varphi\natural_{k+1}(u)\big) \quad \text{by (5.1.4.1)} \\
&= c\big(\eta_{S,A}(\varphi(t))\big).
\end{aligned}
$$

Hence

$$
\begin{aligned}
(5.1.4.3) &= \{\varphi(t) \in \hat{S}_A^{\mathcal{L}} : c\big(\eta_{S,A}(\varphi(t))\big) = A_n\} \\
&= \{s \in \hat{S}_A^{\mathcal{L}} : c\big(\eta_{S,A}(s)\big) = A_n\}
\end{aligned}
$$

and so

$$
\begin{aligned}
&|\varphi(\{t \in F_A(\overline{R}_{k+1} = \overline{S}_{k+1}) : c(t) = A_n\})\ | \\
&= |\ \{s \in \hat{S}_A^{\mathcal{L}} : c(\eta_{S,A}(s)) = A_n\}\ | \\
&= c_n(R_{k+1} = S_{k+1}) \qquad \text{by 5.1.10} \\
&= |\ \{t \in F_A(R_{k+1} = S_{k+1}) : c(t) = A_n\}\ | .
\end{aligned}
$$

Thus φ is injective and so $\hat{S}_A^{\mathcal{L}} \simeq F_A(R_{k+1} = S_{k+1})$.

The proofs of (3)–(5) are similar to this one.

We now prove 5.1.6–5.1.10.

Proof of Lemma 5.1.6. The proof is based on Lemma 4.1.8. The case of $\mathbf{V}(xy = yx)$ will be omitted since it was treated in example 5.1.3. We list the cases that have to be considered.

(1) $S \in \mathbf{V}(\overline{R}_2 = \overline{Q}_2)$;

(2) $S \in \mathbf{V}(ax = axa)$;

(3) $S \in \mathbf{V}(R_k = T_k)$, k odd, $k \geq 3$, $T \in \{Q, S\}$;

(4) $S \in \mathbf{V}(\overline{R}_k = \overline{T}_k)$, k even, $k \geq 3$, $T \in \{Q, S\}$;

(5) $S \in \mathbf{B}$.

We now prove (1)–(5).

(1) Let $S \in \mathbf{V}(\overline{R}_2 = \overline{Q}_2)$ and let $s_1, s_2, s_3 \in \hat{S}^{\mathcal{L}}$. Then

$$\begin{aligned}
\eta_S(s_1 s_3 s_1 s_2 s_3) &= \eta_S(s_1)\eta_S(s_3)\eta_S(s_1)\eta_S(s_2)\eta_S(s_3) \\
&= \eta_S(s_1)\eta_S(s_3)\eta_S(s_1)\eta_S(s_3)\eta_S(s_2) \\
&= \eta_S(s_1)\eta_S(s_3)\eta_S(s_2) \\
&= \eta_S(s_1)\eta_S(s_2)\eta_S(s_3) \\
&= \eta_S(s_1 s_2 s_3).
\end{aligned}$$

Hence 4.1.8 yields that $s_1 s_2 s_3 = s_1 s_3 s_1 s_2 s_3$, thus $\hat{S}^{\mathcal{L}} \in \mathbf{V}(\overline{R}_3 = \overline{Q}_3)$, as required.

(2) Let $S \in \mathbf{V}(ax = axa)$ and let $s_1, s_2, s_3 \in \hat{S}^{\mathcal{L}}$. Then

$$\begin{aligned}
\eta_S(s_1 s_2 s_1 s_3 s_1 s_2 s_3) &= \eta_S(s_1)\eta_S(s_2)\eta_S(s_1)\eta_S(s_3)\eta_S(s_1)\eta_S(s_2)\eta_S(s_3) \\
&= \eta_S(s_1)\eta_S(s_2)\eta_S(s_3)\eta_S(s_1)\eta_S(s_2)\eta_S(s_3) \\
&= \eta_S(s_1)\eta_S(s_2)\eta_S(s_3) \\
&= \eta_S(s_1 s_2 s_3).
\end{aligned}$$

Hence 4.1.8 yields that $s_1 s_2 s_3 = s_1 s_2 s_1 s_3 s_1 s_2 s_3$, thus $\hat{S}^{\mathcal{L}} \in \mathbf{V}(\overline{R}_3 = \overline{S}_3)$, as required.

(3) Let $S \in \mathbf{V}(R_k = S_k)$, k odd, $k \geq 3$. By definition $R_{k+1} = R_k x_{k+1}$ and $S_{k+1} = S_k x_{k+1} R_{k+1}$. We have

$$\begin{aligned}
\eta_S(S_k x_{k+1} R_{k+1}) &= \eta_S(S_k)\eta_S(x_{k+1})\eta_S(R_{k+1}) \\
&= \eta_S(R_k)\eta_S(x_{k+1})\eta_S(R_{k+1}). \qquad (5.1.6.1)
\end{aligned}$$

Indeed, if $S_k = S_k(x_1, \dots, x_k)$, then

$$\begin{aligned}\eta_S(S_k) &= S_k(\eta_S(x_1), \dots, \eta_S(x_k)) \\ &= R_k(\eta_S(x_1), \dots, \eta_S(x_k)) \\ &= \eta_S(R_k).\end{aligned}$$

Hence, using (5.1.6.1), we have

$$\begin{aligned}\eta_S(S_{k+1}) &= \eta_S(R_k x_{k+1})\eta_S(R_{k+1}) \\ &= \eta_S(R_{k+1})\eta_S(R_{k+1}) \\ &= \eta_S(R_{k+1}).\end{aligned}$$

Thus 4.1.8 yields that $S_{k+1} = R_{k+1}$ and so $\hat{S}^{\mathcal{L}} \in \mathbf{V}(R_{k+1} = S_{k+1})$, as required.

Similarly, if $S \in \mathbf{V}(R_k = Q_k)$, k odd, $k \geq 3$, then $\hat{S}^{\mathcal{L}} \in \mathbf{V}(R_{k+1} = Q_{k+1})$.

(4) The proof is similar to the proof of (3).

(5) It is an immediate consequence of 4.1.11.

Proof of Lemma 5.1.7. Let $u, v \in A^+$ be such that $c(u) \supseteq c(v)$, $|c(u)| = |c(v)| + 1$. We prove (i) and (ii).

(i) By 1.5.1 and 2.2.10, if $V = \mathbf{V}(\overline{R}_2 = \overline{Q}_2)$, then

$$u \leq_{\mathcal{L}} v \Longleftrightarrow \begin{cases} c(uv) = c(u) \\ h_2(uv) = h_2(u) \end{cases}$$

and if $V = \mathbf{V}(ax = axa)$, then

$$u \leq_{\mathcal{L}} v \Longleftrightarrow i_2(uv) = i_2(u).$$

But $c(u) \supseteq c(v)$ implies $c(uv) = c(u)$, $h_2(uv) = h_2(u)$, $i_2(uv) = i_2(u)$. Hence in both cases $u \leq_{\mathcal{L}} v$. If $v \leq_{\mathcal{L}} u$, we would get $c(vu) = c(v)$ and this contradicts the fact that $c(u) \neq c(v)$. Therefore $u <_{\mathcal{L}} v$.

(ii) Let $T \in \{Q, S\}$, $k \geq 3$, $V = \mathbf{V}(R_k = T_k)$ for k odd and $V = \mathbf{V}(\overline{R}_k = \overline{T}_k)$ for k even. Then

$$u \leq_{\mathcal{L}} v \iff t_k(uv) = t_k(u) \quad (t = i \text{ if } T = S; t = h \text{ if } T = Q).$$

If $V = \mathbf{V}(R_3 = Q_3)$, we get

$$\begin{aligned} u \leq_{\mathcal{L}} v &\iff h_3(uv) = h_3(u) \\ &\iff \begin{cases} h_2(s(uv)) = h_2(u) \\ \sigma(uv) = \sigma(u) \\ \overline{h}_2(uv) = \overline{h}_2(u) \end{cases} \\ &\iff \overline{h}_2(u) = \overline{h}_2(v) \end{aligned}$$

In the remainder cases we get

$$u \leq_{\mathcal{L}} v \iff \begin{cases} t_k(s(uv)) = t_k(s(u)) \\ \sigma(uv) = \sigma(u) \\ \varepsilon(uv) = \varepsilon(u) \\ \overline{t}_{k-1}(e(uv)) = \overline{t}_{k-1}(e(u)) \end{cases} \qquad \text{by 2.2.9}$$

$$\begin{aligned} &\iff \overline{t}_{k-1}(e(uv)) = \overline{t}_{k-1}(e(u)), \quad \text{since } c(u) \supseteq c(v) \\ &\iff \overline{t}_{k-1}(e(u)v) = \overline{t}_{k-1}(e(u)), \quad \text{since } c(u) \neq c(v) \\ &\iff \overline{t}_{k-1}(e(u))\overline{t}_{k-1}(v) = \overline{t}_{k-1}(e(u)) \\ &\iff \overline{t}_{k-1}(e(u)) \leq_{\mathcal{L}} \overline{t}_{k-1}(v) \end{aligned}$$

since $c(e(u)) = c(v)$, $e(u)$ and v are $\mathcal{J}$-related. (See 2.2.16.) So 1.5.6 yields that

$$u \leq_{\mathcal{L}} v \iff \overline{t}_{k-1}(e(u))\mathcal{L}\overline{t}_{k-1}(v).$$

In particular, if $V = \mathbf{V}(R_3 = S_3)$, we get

$$\begin{aligned} u \leq_{\mathcal{L}} v &\iff \overline{i}_2(e(u))\mathcal{L}\overline{i}_2(v) \\ &\iff \overline{i}_2(e(u)) = \overline{i}_2(v) \end{aligned}$$

since in this case $\mathcal{L}_V = \mathrm{id}_{F_A(xa=axa)}$. (See 3.1.9.)

Proof of Proposition 5.1.8. (i) Let

$$X = \{(s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) : n \geq 1, s_i \in A^+, |\, c(s_i)\, | = i, i = 1, \ldots, n\}.$$

If $V = \mathbf{V}(\overline{R}_2 = \overline{Q}_2)$ or $V = \mathbf{V}(ax = axa)$, the proof of the inclusion $\hat{S}_A^{\mathcal{L}} \subseteq X$ is analogous with the proof made in 5.1.3, for $V = \mathbf{V}(xy = yx)$. We now prove the converse inclusion.

Let $V = \mathbf{V}(\overline{R}_2 = \overline{Q}_2)$ and let $s \in X$. If $s = (s_1), |c(s_1)| = 1$, then $s_1 \in A$ and $s \in \hat{S}_A^{\mathcal{L}}$. Notice that we are identifying s_i with $[s_i]_{\nu(V)}$ and that $s_i \nu s_i^m$, for all $m \geq 1$. (See 2.2.4.)

Let $n \geq 1$ and suppose that all sequences $(s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1)$ of X are in $\hat{S}_A^{\mathcal{L}}$. Let $s = (s_{n+1} <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) \in X$ and let $c(s_{n+1}) \setminus c(s_n) = \{a_{n+1}\}$. Then $s_{n+1} = u_1 a_{n+1} u_2$, with $u_1, u_2 \in A^*$. If $u_1 \in A^+$, then $u_1 = b_1 \ldots b_k$, where $b_i \in A, i = 1, \ldots, k$ and

$$\begin{aligned}
&(b_1)\ldots(b_k)(a_{n+1})(s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1)\\
&= \mathrm{Red}(b_1 \ldots b_k a_{n+1} s_n \leq_{\mathcal{L}} b_2 \ldots b_k a_{n+1} s_n \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} a_{n+1} s_n <_{\mathcal{L}} s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1)\\
&= (b_1 \ldots b_k a_{n+1} s_n <_{\mathcal{L}} s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) \qquad (5.1.8.1)
\end{aligned}$$

by 5.1.2, since

$$c(b_1 \ldots b_k a_{n+1} s_n) = c(a_{n+1} s_n).$$

Hence

$$\begin{aligned}
(5.1.8.1) &= (u_1 a_{n+1} s_n <_{\mathcal{L}} s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1)\\
&= (s_{n+1} <_{\mathcal{L}} s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1)
\end{aligned}$$

since

$$h_2(s_{n+1}) = h_2(u_1 a_{n+1} u_2) = h_2(u_1 a_{n+1} s_n)$$

and

$$c(s_{n+1}) = c(u_1 a_{n+1} s_n)$$

(i.e. $[s_{n+1}]_{\nu(\overline{R}_2=\overline{Q}_2)} = [u_1 a_{n+1} s_n]_{\nu(\overline{R}_2=\overline{Q}_2)}$).

Therefore, by the induction hypothesis, $s \in \hat{S}_A^{\mathcal{L}}$. Thus $X = \hat{S}_A^{\mathcal{L}}$.

Suppose now that $V = \mathbf{V}(ax = axa)$ and let $s \in X$.

If $s = (s_1), |\, c(s_1)\,| = 1$, then $s_1 \in A$ and $s \in \hat{S}_A^{\mathcal{L}}$.

Let $n \geq 1$ and suppose that all sequences $(s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1)$ of X are in $\hat{S}_A^{\mathcal{L}}$. Let $s = (s_{n+1} <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) \in X$ and let $c(s_{n+1}) \setminus c(s_n) = \{a_{n+1}\}$. We check first that

$$i_2(s_{n+1}) = i_2\big(s(s_{n+1})\sigma(s_{n+1})a_{n+1} s_n\big) \tag{5.1.8.2}$$

Indeed, we have

$$\begin{aligned} & i_2\big(s(s_{n+1})\sigma(s_{n+1})a_{n+1} s_n\big) \\ & = i_2[s\big(s(s_{n+1})\sigma(s(s_{n+1}))a_{n+1} s_n\big)]\sigma[s(s_{n+1})\sigma(s_{n+1})a_{n+1} s_n] \qquad \text{by 2.2.3} \\ & = i_2\big(s(s_{n+1})\big)\sigma(s_{n+1}) \\ & = i_2(s_{n+1}). \end{aligned}$$

Now let $s(s_{n+1}) = b_1 \ldots b_r, b_i \in A, i = 1, \ldots, r$. We will see that

$$s = (b_1)\ldots(b_r)(\sigma(s_{n+1}))(a_{n+1})(s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1).$$

Indeed,

$$\begin{aligned} & (b_1)\ldots(b_r)(\sigma(s_{n+1}))(a_{n+1})(s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) \\ & = \mathrm{Red}\,(b_1 \ldots b_r\sigma(s_{n+1})a_{n+1} s_n \leq_{\mathcal{L}} b_2 \ldots b_r\sigma(s_{n+1})a_{n+1} s_n <_{\mathcal{L}} s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) \\ & = (b_1 \ldots b_r\sigma(s_{n+1})a_{n+1} s_n <_{\mathcal{L}} s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) \end{aligned} \tag{5.1.8.3}$$

since $c(b_1 \dots b_r \sigma(s_{n+1}) a_{n+1} s_n) = c(a_{n+1} s_n)$. Thus

$$(5.1.8.3) = \big(s(s_{n+1})\sigma(s_{n+1}) a_{n+1} s_n <_{\mathcal{L}} s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1\big)$$
$$= (s_{n+1} <_{\mathcal{L}} s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) \qquad \text{by (5.1.8.2).}$$

Hence the inclusion hypothesis yields that $s \in \hat{S}_A^{\mathcal{L}}$ and so, $X = \hat{S}_A^{\mathcal{L}}$.

(ii) Let $T \in \{Q, S\}, k \geq 3, V = \mathbf{V}(R_k = T_k)$ for k odd and $V = \mathbf{V}(\overline{R}_k = \overline{T}_k)$ for k even. Let

$$X = \{(s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) : n \geq 1, s_i \in A^+, |c(s_i)| = i, \bar{t}_{k-1}(e(s_i)) = \bar{t}_{k-1}(s_{i-1}), \\ i = 1, \dots, n\}.$$

We prove first the inclusion $\hat{S}_A^{\mathcal{L}} \subseteq X$.

If $a \in A$, then $(a) \in X$. Suppose that $n \geq 1$ and that any product $(a_n) \dots (a_1), a_i \in A, i = 1, \dots, n$, is in X. Let $s = (a_{n+1})(a_n) \dots (a_1)$, $a_i \in A$, $i = 1, \dots, n+1$. Then

$$s = (a_{n+1})[(a_n) \dots (a_1)]$$
$$= (a_{n+1})(s_k <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1)$$

by the induction hypothesis, where $k \geq 1, s_i \in A^+, |c(s_i)| = i, \bar{t}_{k-1}(e(s_i)) = \bar{t}_{k-1}(s_{i-1}), i = 1, \dots, k$. Hence

$$s = \mathrm{Red}(a_{n+1} s_k \leq_{\mathcal{L}} s_k <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1).$$

If $a_{n+1} \in c(s_k)$, then $a_{n+1} s_k \mathcal{L} s_k$ (by 5.1.2) and

$$s = (a_{n+1} s_k <_{\mathcal{L}} s_{k-1} <_{\mathcal{L}} s_k <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1)$$

where $| c(a_{n+1} s_k) | = k$. Moreover in this case

$$\bar{t}_{k-1}(e(a_{n+1} s_k)) = \bar{t}_{k-1}(e(s_k))$$
$$= \bar{t}_{k-1}(s_{k-1}), \qquad \text{by the induction hypothesis}$$

and so $s \in X$.

If $a_{n+1} \notin c(s_k)$ then $a_{n+1}\, s_k <_{\mathcal{L}} s_k$ and

$$s = (a_{n+1}\, s_k <_{\mathcal{L}} s_k <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1)$$

where $|\, c(a_{n+1}\, s_k)\, | = k + 1$. Moreover

$$\overline{t}_{k-1}(e(a_{n+1}\, s_k)) = \overline{t}_{k-1}(s_k)$$

since in this case $\varepsilon(a_{n+1}\, s_k) = a_{n+1}$ and $e(a_{n+1}\, s_k) = s_k$.

Hence $\hat{S}_A^{\mathcal{L}} \subseteq X$.

Conversely, let $s \in X$. If $s = (s_1)$, with $|\, c(s_1)\, | = 1$, then $s_1 \in A$ and $s \in \hat{S}_A^{\mathcal{L}}$. Let $n \geq 1$ and suppose that all sequences $(s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1)$ of X are in $\hat{S}_A^{\mathcal{L}}$. Let $s = (s_{n+1} <_{\mathcal{L}} s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) \in X$.

Since $\overline{t}_{k-1}(e(s_{n+1})) = \overline{t}_{k-1}(s_n)$, then $c(e(s_{n+1})) = c(s_n)$ and we deduce that

$$\{\varepsilon(s_{n+1})\} = c(s_{n+1}) \setminus c(s_n).$$

We check now that

$$t_k(s_{n+1}) = t_k\big(s(s_{n+1})\sigma(s_{n+1})\varepsilon(s_{n+1})s_n\big) \tag{5.1.8.4}$$

Indeed,

$$\begin{aligned}
& t_k\big(s(s_{n+1})\sigma(s_{n+1})\varepsilon(s_{n+1})s_n\big) \\
& \quad = t_k[s\big(s(s_{n+1})\sigma(s_{n+1})\varepsilon(s_{n+1})s_n\big)]\sigma\big(s(s_{n+1})\sigma(s_{n+1})\varepsilon(s_{n+1})s_n\big) \\
& \quad \varepsilon\big(s(s_{n+1})\sigma(s_{n+1})\varepsilon(s_{n+1})s_n\big)\overline{t}_{k-1}[e\big(s(s_{n+1})\sigma(s_{n+1})\varepsilon(s_{n+1})s_n\big)] \\
& \quad = t_k\big(s(s_{n+1})\big)\sigma(s_{n+1})\varepsilon(s_{n+1})\overline{t}_{k-1}(s_n) \\
& \quad = t_k\big(s(s_{n+1})\big)\sigma(s_{n+1})\varepsilon(s_{n+1})\overline{t}_{k-1}\big(e(s_{n+1})\big) \\
& \quad = t_k(s_{n+1}).
\end{aligned}$$

Now, let $s(s_{n+1}) = b_1 \dots b_r, b_i \in A, i = 1, \dots, r$. We show that

$$s = (b_1) \dots (b_r) \big(\sigma(s_{n+1})\big) \big(\varepsilon(s_{n+1})\big)(s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1).$$

Indeed,

$$\begin{aligned}
&(b_1) \dots (b_r) \big(\sigma(s_{n+1})\big) \big(\varepsilon(s_{n+1})\big)(s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) \\
&= \operatorname{Red} \big(b_1 \dots b_r \sigma(s_{n+1}) \varepsilon(s_{n+1}) s_n \leq_{\mathcal{L}} b_2 \dots b_r \sigma(s_{n+1}) \varepsilon(s_{n+1}) s_n \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} \\
&\qquad \sigma(s_{n+1}) \varepsilon(s_{n+1}) s_n \leq_{\mathcal{L}} \varepsilon(s_{n+1}) s_n <_{\mathcal{L}} s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1\big) \\
&= (b_1 \dots b_r \sigma(s_{n+1}) \varepsilon(s_{n+1}) s_n <_{\mathcal{L}} s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) \qquad (5.1.8.5)
\end{aligned}$$

by 5.1.2, since

$$c(b_1 \dots b_r \sigma(s_{n+1}) \varepsilon(s_{n+1}) s_n) = c(\varepsilon(s_{n+1}) s_n).$$

Hence

$$\begin{aligned}
(5.1.8.5) &= \big(s(s_{n+1}) \sigma(s_{n+1}) \varepsilon(s_{n+1}) s_n <_{\mathcal{L}} s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1\big) \\
&= (s_{n+1} <_{\mathcal{L}} s_n <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) \qquad \text{by } (5.1.8.4)
\end{aligned}$$

Therefore, by the induction hypothesis, $s \in \hat{S}_A^{\mathcal{L}}$. Thus $X = \hat{S}_A^{\mathcal{L}}$.

Proof of Lemma 5.1.9. (i) Let V be one of the varieties $\mathbf{V}(\overline{R}_2 = \overline{Q}_2), \mathbf{V}(ax = axa)$ and let $S = F_A(V)$. If $| c(u) | = 1$, then the statement is trivially true, since $| c_1(V) | = 1$, for all $V \in \mathbf{LB_0}$. Suppose the statement holds for $n \geq 1$ and let $u \in A^+$ be such that $| c(u) | = n + 1$.

Fixed $z \in A^+$ such that $| c(z) | = n$, we have

$$\begin{aligned}
&|(u \leq_{\mathcal{L}} s_n \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} s_1) : s_i \in A^+, | c(s_i) | = i, i = 1, \dots, n\}| \\
&= |\{v \in A^+ : u \leq_{\mathcal{L}} v, | c(v) | = n\} | \,.\, | \{(z \leq_{\mathcal{L}} s_{n-1} \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} s_1) : s_i \in A^+, | c(s_i) | = i, \\
&\qquad i = 1, \dots, n-1\}| \qquad (5.1.9.1)
\end{aligned}$$

If $V = \mathbf{V}(\overline{R}_2 = \overline{Q}_2)$ then

$$|\{v : u \leq_{\mathcal{L}} v, |c(v)| = n\}| = (n+1)c_n(\overline{R}_2 = \overline{Q}_2),$$

since there are $n+1$ subsets of $c(u)$ with n elements. On the other hand the induction hypothesis yields in this case that

$$|\{(z \leq_{\mathcal{L}} s_{n-1} \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} s_1) : s_i \in A^+, |c(s_i)| = i, i = 1, \ldots, n-1\}| = \frac{c_n(\overline{R}_3 = \overline{Q}_3)}{c_n(\overline{R}_2 = \overline{Q}_2)}.$$

Hence

$$\begin{aligned}(5.1.9.1) &= (n+1)c_n(\overline{R}_2 = \overline{Q}_2)\frac{c_n(\overline{R}_3 = \overline{Q}_3)}{c_n(\overline{R}_2 = \overline{Q}_2)} \\ &= (n+1)c_n(\overline{R}_3 = \overline{Q}_3) \\ &= \frac{c_{n+1}(\overline{R}_3 = \overline{Q}_3)}{c_{n+1}(\overline{R}_2 = \overline{Q}_2)} \qquad \text{by 3.1.10.}\end{aligned}$$

If $V = \mathbf{V}(ax = axa)$ then

$$|\{v : u \leq_{\mathcal{L}} v, |c(v)| = n\}| = (n+1)c_n(ax = axa).$$

and the induction hypothesis yields that

$$|\{(z \leq_{\mathcal{L}} s_{n-1} \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} s_1) : s_i \in A^+, |c(s_i)| = i, i = 1, \ldots, n-1\}| = \frac{c_n(\overline{R}_3 = \overline{S}_3)}{c_n(ax = axa)}$$

Hence

$$\begin{aligned}(5.1.9.1) &= (n+1)c_n(ax = axa)\frac{c_n(\overline{R}_3 = \overline{S}_3)}{c_n(ax = axa)} \\ &= (n+1)c_n(\overline{R}_3 = \overline{S}_3) \\ &= \frac{c_{n+1}(\overline{R}_3 = \overline{S}_3)}{c_{n+1}(ax = axa)} \qquad \text{by 3.1.10.}\end{aligned}$$

(ii) Let $T \in \{Q, S\}$, $k \geq 3$, k odd and let $V = \mathbf{V}(R_k = T_k)$.

If $|c(u)| = 1$, the statement is trivially true. Suppose the statement holds for $n \geq 1$ and let $u \in A^+$ be such that $|c(u)| = n+1$. Fixed $z \in A^+$ such that $|c(z)| = n$, we have

$$\begin{aligned}
&|(u \leq_{\mathcal{L}} s_n \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} s_1) : s_i \in A^+, |c(s_i)| = i, \bar{t}_{k-1}(e(u)) = \bar{t}_{k-1}(v),\\
&\qquad t_{k-1}(e(s_i)) = \bar{t}_{k-1}(s_{i-1}), i = 1, \ldots, n\} |\\
&=| \{v : u \leq_{\mathcal{L}} v, | c(v) |= n, \bar{t}_{k-1}(e(u)) = \bar{t}_{k-1}(v)\} | \,.\\
&.\, | \{(z \leq_{\mathcal{L}} s_{n-1} \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} s_1) : s_i \in A^+, | c(s_i) |= i, \bar{t}_{k-1}(e(s_i)) = \bar{t}_{k-1}(s_{i-1}),\\
&\qquad i = 1, \ldots, n-1, \bar{t}_{k-1}(e(z)) = \bar{t}_{k-1}(s_{n-1})\} | \qquad (5.1.9.2)
\end{aligned}$$

But

$$\begin{aligned}
&|\{v : u \leq_{\mathcal{L}} v, | c(v) |= n, \bar{t}_{k-1}(e(u)) = \bar{t}_{k-1}(v)\} |\\
&=| \{v : \bar{t}_{k-1}(e(u))\mathcal{L}\bar{t}_{k-1}(v), \bar{t}_{k-1}(e(u)) = \bar{t}_{k-1}(v)\} | \qquad \text{by 5.1.7}\\
&=| \{v : \bar{t}_{k-1}(e(u)) = \bar{t}_{k-1}(v)\} |\\
&= \frac{c_n(R_k = T_k)}{c_n(R_{k-1} = T_{k-1})}
\end{aligned}$$

since there are $c_n(R_k = T_k)$ words $t_k(v)$ with $|c(v)| = n$ and there are $c_n(R_{k-1} = T_{k-1})$ words $\bar{t}_{k-1}(w)$ with $|c(w)| = n$ ($\bar{t}_{k-1}(e(u))$ is such a word).

On the other hand, the induction hypothesis yields that

$$\begin{aligned}
&|\{(z \leq_{\mathcal{L}} s_{n-1} \leq_{\mathcal{L}} \cdots \leq_{\mathcal{L}} s_1) : s_i \in A^+, | c(s_i) |= i, \bar{t}_{k-1}(e(s_i)) = \bar{t}_{k-1}(s_{i-1}),\\
&\qquad i = 1, \ldots, n-1, \bar{t}_{k-1}(e(z)) = \bar{t}_{k-1}(s_{n-1})\}|\\
&= \frac{c_n(R_{k+1} = T_{k+1})}{c_n(R_k = T_k)}.
\end{aligned}$$

Hence

$$\begin{aligned}
(5.1.9.2) &= \frac{c_n(R_k = T_k)}{c_n(R_{k-1} = T_{k-1})} \cdot \frac{c_n(R_{k+1} = T_{k+1})}{c_n(R_k = T_k)}\\
&= \frac{c_{n+1}(R_{k+1} = T_{k+1})}{c_{n+1}(R_k = T_k)} \qquad \text{by 3.1.10.}
\end{aligned}$$

For $V = \mathbf{V}(\overline{R}_k = \overline{T}_k)$, $k \geq 3$, k even, the proof is similar.

Proof of Lemma 5.1.10. The proof results immediately from 5.1.9. Indeed, if $s = (s_n <_{\mathcal{L}} s_{n-1} <_{\mathcal{L}} \cdots <_{\mathcal{L}} s_1) \in \hat{S}^{\mathcal{L}}_A$, then $\eta_{S,A}(s) = s_n$ and the number of elements s_n such that $c(s_n) = A_n$ is

$$
\begin{aligned}
&c_n(ax = axa) \ \text{ if } \ V = \mathbf{V}(ax = axa),\\
&c_n(R_k = T_k) \ \text{ if } \ V = \mathbf{V}(R_k = T_k), T \in \{Q, S\}, k \geq 3, k \text{ odd},\\
&c_n(\overline{R}_k = \overline{T}_k) \ \text{ if } \ V = \mathbf{V}(\overline{R}_k = \overline{T}_k), T \in \{Q, S\}, k \geq 3, k \text{ even}.
\end{aligned}
$$

2. Rhodes expansions of the free objects in varieties of band monoids

In 4.2 we introduced the notion of Rhodes expansion of a monoid. The natural isomorphism established in Theorem 4.2.5 will enable us to deduce, for band monoids, results corresponding to the ones obtained for bands in the previous section.

Proposition 5.2.1. *If* $V \in \mathbf{LBM_0}$ *and* $M = F_A(V)$, *then*

$$\hat{M}^{\mathcal{L}}_{e,A} \simeq F_A(V^r).$$

Proof. Let $V \in \mathbf{LBM_0}$ and $M = F_A(V)$. We have shown in 3.3.1 that $M \simeq T^{\mathrm{I}}$, where $T = F_A(< V >_{\mathbf{S}})$ and $< V >_{\mathbf{S}}$ is the Birkhoff variety of bands generated by V. Then

$$
\begin{aligned}
\hat{M}^{\mathcal{L}}_{e,A} &\simeq (\hat{T}^{\mathrm{I}})^{\mathcal{L}}_{e,A}\\
&\simeq (\hat{T}^{\mathcal{L}}_A)^{\mathrm{I}} \quad \text{by 4.2.5.}
\end{aligned}
$$

Now, by 5.1.4, $\hat{T}^{\mathcal{L}}_A \simeq F_A(< V >^{r}_{S})$. Hence,

$$\begin{aligned} \hat{M}^{\mathcal{L}}_{e,A} &\simeq [F_A(< V >^{r}_{S})]^{I} \\ &= [F_A(< V^{r} >_{S})]^{I} \quad \text{since} < V >^{r}_{S} = < V^{r} >_{S} \text{ (by 2.3.4)} \\ &\simeq F_A(V^{r}) \quad \text{by 3.3.1.} \end{aligned}$$

that is, $\hat{M}^{\mathcal{L}}_{e,A} \simeq F_A(V^{r})$.

Definition 5.2.2. A variety V of bands [band monoids] is said to be *closed for Rhodes expansions* if for any semigroup [monoid] S in V, $\hat{S}^{\mathcal{L}}_A$ and $\hat{S}^{\mathcal{R}}_A$ [$\hat{S}^{\mathcal{L}}_{e,A}$ and $\hat{S}^{\mathcal{R}}_{e,A}$] are in V.

Theorem 5.2.3. *The only varieties of band monoids closed for Rhodes expansions are the trivial variety* $\mathbf{VM}(x = 1)$ *and the variety* $\mathbf{BM}$ *of all band monoids.*

Proof. Let $V \in \mathbf{LBM_0}$, $V \neq \mathbf{BM}$. Then $V \neq V^{l}$ or $V \neq V^{r}$.

Let $M = F_A(V)$. Proposition 5.2.1 asserts that $\hat{M}^{\mathcal{L}}_{e,A} \simeq F_A(V^{r})$ and dually, $\hat{M}^{\mathcal{R}}_{e,A} \simeq F_A(V^{l})$. Hence $\hat{M}^{\mathcal{L}}_{e,A} \in V^{r}$ and $\hat{M}^{\mathcal{R}}_{e,A} \in V^{l}$.

If $V \neq V^{r}$ then for $|A|$ big enough $F_A(V^{r}) \neq F_A(V)$ (by 3.2.8). Thus $\hat{M}^{\mathcal{L}}_{e,A} \notin V$. If $V \neq V^{l}$ then for $|A|$ big enough $F_A(V^{l}) \neq F_A(V)$ and $\hat{M}^{\mathcal{R}}_{e,A} \notin V$.

Remark 5.2.4. Theorem 5.2.3 is not true for varieties of bands. (See 5.1.1.)

Remark 5.2.5. Theorem 5.2.3 also holds for Eilenberg varieties of band monoids (defined next chapter). Notice that if S is a finite semigroup, its Rhodes expansion is also finite.

PART III – THE VARIETIES OF LANGUAGES

CORRESPONDING TO

THE VARIETIES OF FINITE BAND MONOIDS

CHAPTER 6. VARIETIES OF LANGUAGES

In this chapter we present various notions and results respecting the theory of formal languages. Here we consider only finite alphabets.

In the first section we focus on the Variety Theorem, which makes the link between the notions of variety of formal languages, presented here and variety of finite semigroups, presented in 2.4.

In the second section we present some specific results on varieties of monoids and the corresponding varieties of languages, which we shall consistently use in the remainder. We give some results concerning the varieties of languages corresponding to the varieties of finite band monoids, depending on the cardinality of the alphabet. These are general results, not yet concerning the direct description of those varieties of languages, but necessary for doing it.

1. Varieties of languages. Eilenberg's Variety Theorem

In this section A will denote a finite alphabet. A *language* on the alphabet A is a subset of the free monoid A^*.

If L is a language, the *dual* of L is the language, denoted by $\overline{L}$, obtained from L by taking the duals of the words in L.

We shall use the following operations over languages:

Definition 6.1.1. Let L, K be languages of A^*.

(i) The *star* of L, denoted by L^*, is the submonoid of A^* generated by L.

(ii) The *product* of L by K is the language

$$LK = \{uv \in A^* : u \in L, v \in K\}$$

(iii) The *quotient* of L by K is the language

$$K^{-1}L = \{v \in A^* : Kv \cap L \neq \emptyset\}$$

and dually,

$$LK^{-1} = \{v \in A^* : vK \cap L \neq \emptyset\}.$$

If $u \in A^*$, we often identify the word u with the language $\{u\}$. For instance, we denote by $u^{-1}L$ the language

$$\{u\}^{-1}L = \{v \in A^* : uv \in L\}$$

We shall also use the classical boolean operations of finite union, finite intersection and complementation.

Definition 6.1.2. Let M be a monoid and let $\varphi : A^* \to M$ be a monoid morphism. We say that a language $L \subseteq A^*$ is *recognized* by φ if there is $P \subseteq M$ such that $L = \varphi^{-1}(P)$. We also say that M *recognizes* L.

Definition 6.1.3. A language is called *recognizable* if it is recognized by a finite monoid.

We now give Eilenberg's definition of a variety of languages.

Definition 6.1.4. A *variety of languages* is a correspondence $\mathcal{V}$ that associates to each finite alphabet A a set $A^*\mathcal{V}$ of recognizable languages such that

(1) for each alphabet A, $A^*\mathcal{V}$ is a boolean algebra (with operations union and complementation),

(2) if $\varphi : A^* \to B^*$ is a free monoid morphism, $L \in B^*\mathcal{V}$ implies $\varphi^{-1}(L) \in A^*\mathcal{V}$,

(3) if $L \in A^*\mathcal{V}$ and $a \in A$, then $a^{-1}L, La^{-1} \in A^*\mathcal{V}$.

Definition 6.1.5. Given two varieties of languages $\mathcal{V}$ and $\mathcal{W}$, we say that $\mathcal{V} \subseteq \mathcal{W}$ if, for every alphabet A, $A^*\mathcal{V} \subseteq A^*\mathcal{W}$.

Proposition 6.1.6. *The relation $\subseteq$ introduces a structure of complete lattice in the class of all varieties of languages.*

This proposition is an immediate consequence of the following lemma:

Lemma 6.1.7. *Given a family $\{\mathcal{V}_i\}_{i\in I}$ of varieties of languages, for each alphabet A define*

$$\mathcal{V}(A) = \bigvee_{i\in I} A^*\mathcal{V}_i \quad ; \quad \mathcal{W}(A) = \bigwedge_{i\in I} A^*\mathcal{V}_i$$

where $\bigvee_{i\in I} A^*\mathcal{V}_i[\bigwedge_{i\in I} A^*\mathcal{V}_i]$ is the supremum [infimum] of the boolean subalgebras $A^*\mathcal{V}_i$ of $\mathcal{P}(A^*)$. Then

(i) *The mappings $A \longmapsto \mathcal{V}(A)$ and $A \longmapsto \mathcal{W}(A)$ are varieties of languages.*

(ii) *The variety of languages $A^* \longmapsto \mathcal{V}(A)$ is the supremum of the family $\{V_i\}_{i\in I}$; The variety of languages $A^* \longmapsto \mathcal{W}(A)$ is the infimum of the family $\{V_i\}_{i\in I}$.*

Proof. (i) It is an immediate consequence of the following fact:

$\bigvee_{i\in I} A^*\mathcal{V}_i$ is equal to the following set of subsets of $\mathcal{P}(A^*)$

$$\left\{ \bigcup_{1\leq j\leq s} \bigcap_{1\leq r\leq r_j} X_{j,r} : X_{j,r} \in \bigcup_{i\in I} A^*\mathcal{V}_i \text{ or } A^*\backslash X_{j,r} \in \bigcup_{i\in I} A^*\mathcal{V}_i, s, r_j \in \mathbb{N}, j = 1,\ldots,s \right\}$$

(ii) It is an immediate consequence of (i) and the fact that $\mathcal{P}(A^*)$ is a complete lattice.

We notice that the relation $\subseteq$ defines a structure of complete lattice in the class of Eilenberg varieties of monoids.

We now present Eilenberg's variety theorem, which enables us to associate with each variety of finite monoids a certain class of recognizable languages.

Theorem 6.1.8. *Let Γ be the correspondence from the varieties of finite monoids into the varieties of languages defined in the following way: if V is a variety of monoids, then $\Gamma(V) = \mathcal{V}$, where $\mathcal{V}$ is the variety of languages defined by*

$$\begin{aligned} \mathcal{V}(A) &= A^*\mathcal{V} \\ &= \textit{the set of languages in } A^* \textit{ recognized by monoids in } V. \end{aligned}$$

Then Γ is an isomorphism of complete lattices.

In the remainder, by "the variety of languages corresponding to a variety of finite monoids V" we mean the image of V by the function Γ defined above.

2. Some results on varieties of languages

In this section A will denote a finite alphabet.

When a variety of finite monoids is generated by a single monoid, as shall happen with the varieties of finite band monoids, we have a direct description of the corresponding languages. Indeed, we shall consistently use the following result:

Proposition 6.2.1 [20]. *Let V be a variety of finite monoids generated by a monoid M and let $\mathcal{V}$ be the corresponding variety of languages. Then, for each alphabet A, $A^*\mathcal{V}$ is the boolean algebra generated by the languages of the form $\varphi^{-1}(m)$, where $\varphi : A^* \to M$ is an arbitrary morphism and $m \in M$.*

We now state some results concerning the varieties of languages corresponding to the varieties of finite band monoids.

Notation 6.2.2. We denote by $\mathcal{V}(P = Q)$ the variety of languages corresponding to the variety of finite band monoids $\mathbf{VM}(P = Q)$.

Also, $A^*(P = Q)$ will denote the image of the alphabet A by the function $\mathcal{V}(P = Q)$. In the particular case of the variety **FBM** of all finite band monoids, we denote by $\mathcal{FBM}$ the corresponding variety of languages.

If $\mathcal{V}$ is the variety of languages corresponding to a variety of finite band monoids V, since A is finite the cardinality of $A^*\mathcal{V}$ is determined by the cardinality of the free object $F_A(V)$. Indeed, we have

Proposition 6.2.3. *Let V be a variety of finite band monoids and let $\mathcal{V}$ be the corresponding variety of languages. Then, the elements of $A^*\mathcal{V}$ are the inverse images of the subsets of $F_A(V)$ by the canonical epimorphism $\natural_V$.*

Moreover $A^*\mathcal{V}$ is a finite set with cardinality $2^{|F_A(V)|}$.

Proof. By Eilenberg's variety theorem, the languages of $A^*\mathcal{V}$ are of the kind $\varphi^{-1}(X)$, where $X \subseteq M$, for some $M \in V$ and some morphism from A^* into M. In particular, the languages $\natural_V^{-1}(X)$, $X \subseteq F_A(V)$ are in $A^*\mathcal{V}$. Conversely, if we have a language $\varphi^{-1}(X)$ where φ is a morphism from A^* into M and $X \subseteq M$, let ψ be the unique morphism from $F_A(V)$ into M such that $\psi\natural_V = \varphi$. Then, $\varphi^{-1}(X) = \natural_V^{-1}(\psi^{-1}(X))$, where $\psi^{-1}(X) \subseteq F_A(V)$.

The last statement follows from the fact that the map $\overline{\natural}_V : \mathcal{P}(F_A(V)) \to A^*\mathcal{V}$ defined by

$$\overline{\natural}_V(X) = \natural_V^{-1}(X)$$

is a bijection.

Corollary 6.2.4. *Let V and W be varieties of finite band monoids such that $V \subseteq W$. Then*

$$A^*\mathcal{V} = A^*\mathcal{W} \Longleftrightarrow F_A(V) = F_A(W).$$

Proof. Since $V \subseteq W$, by 6.1.8 we get $A^*\mathcal{V} \subseteq A^*\mathcal{W}$. Thus,

$$\begin{aligned} A^*\mathcal{V} = A^*\mathcal{W} &\Longleftrightarrow |A^*\mathcal{V}| = |A^*\mathcal{W}| \\ &\Longleftrightarrow |F_A(V)| = |F_A(W)| \qquad \text{by 6.2.3} \\ &\Longleftrightarrow F_A(V) = F_A(W). \qquad \text{(See 2.1.20.)} \end{aligned}$$

We finish this section with a result that follows immediately from 6.2.4 and 3.2.5.

Corollary 6.2.5. *Let $n \geq 2$ and let $|A| = N$.*

(i) *If $1 \leq N < n$, then*

$$A^*(R_n = S_n) = A^*(R_{N+1} = S_{N+1}).$$

(ii) *If $N \geq n$, then the following inclusion is strict*

$$A^*(R_n = S_n) \subset A^*(R_{n+1} = S_{n+1}).$$

CHAPTER 7. THE VARIETIES OF LANGUAGES CORRESPONDING TO THE VARIETIES OF FINITE BAND MONOIDS

In the first section we present a family of subdirectly irreducible generators for the varieties of finite band monoids. It was introduced by Gerhard [12] in 1972. We state some results concerning these monoids, which will be useful later.

In the second section we present the main result of this chapter, namely a direct description of the varieties of languages corresponding to the varieties of finite band monoids.

1. A family of subdirectly irreducible generators of the varieties of finite band monoids

In this section, A will denote a finite alphabet. Also, we denote by $[n, n+m]$, $n \geq 0$, $m \geq 1$, the set $\{n, n+1, \ldots, n+m\}$.

We adopt from Gerhard [12] the following construction.

Let $n \geq 2$. Let $Z_n = \{a_i^n : 0 \leq i \leq n-1\}$ where a_i^n, for $0 \leq i \leq n-1$, is the constant map from $[0, n-1]$ into itself with value i.

Let $W_n = \{b_i^n : 1 \leq i \leq n-1\}$ where b_i^n, for $1 \leq i \leq n-1$, is the map from $[0, n-1]$ into itself defined by

$$b_i^n(j) = \begin{cases} 0, & \text{if } 0 \leq j \leq i,\ j \text{ even} \\ & \text{and, if } i \text{ is even, for all } j \geq i\,; \\ & \\ 1, & \text{if } 0 \leq j \leq i,\ j \text{ odd} \\ & \text{and, if } i \text{ is odd, for all } j \geq i. \end{cases}$$

Indeed, b_i^n, $1 \leq i \leq n-1$, calculates parity on $[0, i]$ and is constant on $[i, n-1]$, the constant being determined by the parity of i.

Let id_n be the identity map from $[0, n-1]$ into itself and let $B_n = A_n \cup \{\mathrm{id}_n\}$, where $A_2 = Z_2$, $A_3 = Z_3 \cup W_3$ and A_n, for $n \geq 4$, is defined inductively as follows.

If $n \geq 4$ and α is a map from $[0, n-3]$ into itself, define $\overline{\alpha}$ from $[o, n-1]$ into itself by

$$\overline{\alpha}(0) = 0 \quad , \quad \overline{\alpha}(1) = 1 \quad , \quad \overline{\alpha}(i+2) = \alpha(i) + 2 .$$

For any set L of maps from $[0, n-3]$ into itself, let $L^n = \{\overline{\alpha} : \alpha \in L\}$.

For $n \geq 4$ let $A_n = Z_n \cup W_n \cup A^n_{n-2}$. Indeed, A_n is a semigroup and B_n is a monoid.

For $n \geq 4$, B_n is isomorphic to $Z_n \cup W_n \cup B_{n-2}$, since A^n_{n-2} is isomorphic to A_{n-2} .

Example 7.1.1. We have

$A_5 = Z_5 \cup W_5 \cup A_3^5$, where

$$Z_5 = \left\{ \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 2 & 2 & 2 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 & 4 \end{pmatrix} \right\}$$

$$W_5 = \left\{ \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} \right\}$$

$$A_3 = \left\{ \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 2 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 0 \end{pmatrix} \right\}$$

and

$$A_3^5 = \left\{ \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 4 & 4 & 4 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 & 2 \end{pmatrix} \right\}.$$

We now give a sequence of lemmas concerning the structure of the monoids B_n, $n \geq 2$.

Lemma 7.1.2. *The following statements hold.*

(i) *For $n \geq 2$, Z_n is an ideal of B_n. The elements of Z_n are left zeros of B_n.*

(ii) *For $n \geq 3$, W_n is a right zero subsemigroup of B_n.*

(iii) *For $n \geq 4$, $W_n \cup A^n_{n-2}$ is a subsemigroup of B_n. The elements of W_n are right zeros of $W_n \cup A^n_{n-2}$.*

Proof. (i) This is obvious, since the elements of Z_n are constant maps.

(ii) This is obvious, since if $n \geq 3$ and $i, j \in [0, n-1]$, then $b^n_j(\gamma) \in \{0, 1\}$, for any $\gamma \in [0, n-1]$ and $b^n_i(0) = 0, b^n_i(1) = 1$.

(iii) Let $n \geq 4$. We notice that the elements of W_n are the only elements φ of B_n such that

$$\varphi(0) = 0 \ , \ \varphi(1) = 1 \text{ and } \operatorname{Im}\varphi \subseteq \{0, 1\}.$$

If $b^n_i \in W_n$ and $\alpha \in A_{n-2}$, then

$$b^n_i \overline{\alpha}(0) = \overline{\alpha} b^n_i(0) = 0$$

and

$$b^n_i \overline{\alpha}(1) = \overline{\alpha} b^n_i(1) = 1,$$

since $\overline{\alpha}(0) = 0$ and $\overline{\alpha}(1) = 1$. Also,

$$\mathrm{Im}(b_i^n\overline{\alpha}) \subseteq \mathrm{Im}\, b_i^n \subseteq \{0,1\}$$

and

$$\mathrm{Im}(\overline{\alpha}b_i^n) \subseteq \overline{\alpha}(\mathrm{Im}\, b_i^n) \subseteq \overline{\alpha}(\{0,1\}) \subseteq \{0,1\}.$$

Thus, $\overline{\alpha}b_i^n, b_i^n\overline{\alpha} \in W_n$.

Clearly, $\overline{\alpha}b_i^n = b_i^n$.

Let $B'_n = B_n \setminus Z_n$, for $n \geq 2$. Notice that

$$|B'_n| = \frac{1}{2}n(n-1)\ ,\ |A_n| = \frac{1}{2}n(n+1) - 1 \ \text{ and } \ |B_n| = \frac{1}{2}n(n+1).$$

Gerhard [**12**] proved the following theorem

Theorem 7.1.3. *Let $n \geq 2$. Then*

(i) *$\overline{B}_n$ is isomorphic to B'_{n+1}.*

(ii) *B_n generates $\mathbf{VM}(R_n = S_n)$, for n odd.*

(iii) *B'_{n+1} generates $\mathbf{VM}(R_n = S_n)$, for n even.*

For $n \geq 2$, we denote by t_n the immersion from B_n into B_{n+2} defined by $t_n(\alpha) = \overline{\alpha}$. The next lemma concerns the behaviour of these morphisms.

Lemma 7.1.4. *Let $n > 2$ and $k \geq 0$.*

(i) *For $i \in [0, n-1]$, the image of a_i^n by the composed map $t_{n+2k} \circ \cdots \circ t_{n+2} \circ t_n$, is the element of B_{n+2k+2} defined by*

$$(t_{n+2k} \circ \cdots \circ t_{n+2} \circ t_n)(a_i^n)(\gamma) = \begin{cases} \gamma, & \text{if } \ 0 \leq \gamma \leq 2(k+1) - 1; \\ i + 2(k+1), & \text{if } \ 2(k+1) \leq \gamma \leq n + 2k + 1. \end{cases}$$

(ii) *For $j \in [1, n-1]$, the image of b_j^n by the composed map $t_{n+2k} \circ \cdots \circ t_{n+2} \circ t_n$ is the element of B_{n+2k+2} defined by*

$$(t_{n+2k} \circ \cdots \circ t_{n+2} \circ t_n)(b_j^n)(\gamma) = \begin{cases} \gamma, & \text{if } 0 \leq \gamma \leq 2(k+1)-1; \\ 2(k+1), & \text{if } 2(k+1) \leq \gamma \leq j+2(k+1), \\ & \gamma \text{ even and, if } j \text{ is even, for all} \\ & \gamma \geq j+2(k+1); \\ 1+2(k+1), & \text{if } 2(k+1) \leq \gamma \leq j+2(k+1), \\ & \gamma \text{ odd and, if } j \text{ is odd, for all} \\ & \gamma \geq j+2(k+1). \end{cases}$$

Proof. The proofs, by induction on k, are routine and are omitted.

Example 7.1.5. We have

$$t_7 \circ t_5(a_3^5) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 1 & 2 & 3 & 7 & 7 & 7 & 7 & 7 \end{pmatrix}$$

and

$$t_7 \circ t_5 \circ t_3(b_2^3) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 6 \end{pmatrix}.$$

In the sequel we denote by $A_n^{n+2k} (k \geq 1)$ the image of A_n by the map $t_{n+2(k-1)} \circ \cdots \circ t_{n+2} \circ t_n$ and we identify $y \in A_n$ with its image by this map.

We now compute, for n odd, some products in B_n, which will be useful later.

Lemma 7.1.6. *Let $n = 2t+1, t \geq 1$ and let $k, l \in [1, t]$.*

$$\text{(i) } a_i^{2k+1} a_j^{2l+1} = \begin{cases} a_i^{2k+1}, & \text{if } l \leq k; \\ a_j^{2l+1}, & \text{if } l > k, 0 \leq j \leq 2(l-k); \\ a_{i+2(l-k)}^{2l+1}, & \text{if } l > k, j \geq 2(l-k), \end{cases}$$

for any $i, j \in [0, n-1]$.

$$\text{(ii) } b_i^{2k+1} b_j^{2l+1} = \begin{cases} b_j^{2l+1}, & \text{if } l \geq k; \\ b_i^{2k+1}, & \text{if } l < k, 1 \leq i \leq 2(k-l); \\ b_{j+2(k-l)}^{2k+1}, & \text{if } l < k, i > 2(k-l), \end{cases}$$

for any $i, j \in [1, n-1]$.

$$\text{(iii) } b_i^{2k+1} a_j^{2l+1} = \begin{cases} b_i^{2k+1}, & \text{if } l < k, i \leq 2(k-l); \\ b_{2(k-l)}^{2k+1}, & \text{if } l < k, i > 2(k-l), \\ & 0 \leq j \leq i - 2(k-l), j \text{ even}, \\ & \text{and, if } i \text{ is even}, \ j \geq i - 2(k-l); \\ b_{2(k-l)-1}^{2k+1}, & \text{if } l < k, i > 2(k-l), \\ & 0 \leq j \leq i - 2(k-l), j \text{ odd}, \\ & \text{and, if } i \text{ is odd}, \ j \geq i - 2(k-l); \\ a_j^{2l+1}, & \text{if } l \geq k, 0 \leq j < 2(l-k); \\ a_{2(l-k)}^{2l+1}, & \text{if } l \geq k, 2(l-k) \leq j \leq i + 2(l-k), j \text{ even} \\ & \text{and, if } i \text{ is even}, \ j \geq i + 2(l-k); \\ a_{2(l-k)+1}^{2l+1}, & \text{if } l \geq k, 2(l-k) \leq j \leq i + 2(l-k), j \text{ odd} \\ & \text{and, if } i \text{ is odd}, \ j \geq i + 2(l-k), \end{cases}$$

for any $i \in [1, n-1]$ *and* $j \in [0, n-1]$.

Proof. (i) Let $i, j \in [0, n-1]$ and suppose that $l \leq k$. Since A_{2k+1}^n is isomorphic to A_{2k+1} and A_{2l+1}^n is isomorphic to A_{2l+1}^{2k+1} (a subsemigroup of A_{2k+1}), we get, by 7.1.2(i)

$$a_i^{2k+1} a_j^{2l+1} = a_i^{2k+1}.$$

For $l > k$, by 7.1.4(i) we get

$$a_i^{2k+1}a_j^{2l+1}(\gamma) = \begin{cases} a_i^{2k+1}(\gamma), & \text{if } 0 \le \gamma \le 2(t-l)-1; \\ a_i^{2k+1}(j+2(t-l)), & \text{if } 2(t-l) \le \gamma \le 2t, \end{cases}$$

$$= \begin{cases} \gamma, & \text{if } 0 \le \gamma \le 2(t-l)-1, \\ & 0 \le \gamma \le 2(t-k)-1; \\ i+2(t-k), & \text{if } 0 \le \gamma \le 2(t-l)-1, \\ & 2(t-k) \le \gamma \le 2t; \\ j+2(t-l), & \text{if } 2(t-l) \le \gamma \le 2t, \\ & 0 \le j+2(t-l) \le 2(t-k)-1; \\ i+2(t-k), & \text{if } 2(t-l) \le \gamma \le 2t, \\ & 2(t-k) \le j+2(t-l) \le 2t; \end{cases}$$

$$= \begin{cases} \gamma, & \text{if } 0 \le \gamma \le 2(t-l)-1; \\ j+2(t-l), & \text{if } 2(t-l) \le \gamma \le 2t, \\ & 0 \le j \le 2(l-k)-1; \\ i+2(t-k), & \text{if } 2(t-l) \le \gamma \le 2t, \\ & 2(l-k) \le j \le 2l, \end{cases}$$

since the case $0 \le \gamma \le 2(t-l)-1, 2(t-k) \le \gamma \le 2t$ cannot occur.

Hence, if $0 \le j < 2(l-k)$, we get

$$a_i^{2k+1}a_j^{2l+1}(\gamma) = \begin{cases} \gamma, & \text{if } 0 \le \gamma \le 2(t-l)-1; \\ j+2(t-l), & \text{if } 2(t-l) \le \gamma \le 2t, \end{cases}$$

$$= a_j^{2l+1}(\gamma).$$

If $j \geq 2(l-k)$, we get

$$a_i^{2k+1}a_j^{2l+1}(\gamma) = \begin{cases} \gamma, & \text{if } 0 \leq \gamma \leq 2(t-l)-1; \\ i+2(t-k), & \text{if } 2(t-k) \leq \gamma \leq 2t, \end{cases}$$

$$= a_{i+2(l-k)}^{2l+1}(\gamma).$$

(ii) The proof goes along the same lines as the proof of part (i), now using 7.1.4(ii).

(iii) The proof goes along the same lines as the proof of part (i), now using both 7.1.4(i), (ii).

With the help of the lemma above, it is easy to complete the tables of the B_N's. The tables of B_3 and B_4 are pictured in Figure 4.

We finish this section with the following simple statement, which will be useful later.

Lemma 7.1.7. *Let $t \geq 1$ and $1 \leq \gamma \leq t$. Then*

(i) $\prod_{i=\gamma}^{t} a_{2i-1}^{2i+1} = a_{2t-1}^{2t+1}$.

(ii) $\prod_{i=0}^{\gamma-1} b_{2(t-i)-1}^{2(t-i)+1} = b_{2t-1}^{2t+1}$.

Proof. The proofs, by induction on t (using 7.1.2 and 7.1.6) are routine and are omitted.

Example 7.1.8. We have

$$a_3^5 a_5^7 a_7^9 = a_7^9$$

and

$$b_9^{11} b_7^9 b_5^7 b_3^5 = b_9^{11}.$$

2. The varieties of languages corresponding to the varieties of finite band monoids

In this section we establish the main theorem of this chapter, which concerns the description of the varieties of languages corresponding to the varieties of finite band monoids.

It is known that for each alphabet A, $A^*(xy = yx)$ and $A^*(ax = axa)$ are respectively the boolean algebra generated by the languages of the form $A_0^*, A_0 \subseteq A$ (see [**20**, page 40]) and the boolean algebra generated by the languages of the form $A^*a_1A_0^*$, where $A_0 \subset A$ and $a_1 \in A$. (See [**21**].) Here we investigate this correspondence for all varieties of **LBM** and our results include these results. Notice that $\mathbf{VM}(R_2 = S_2) = \mathbf{VM}(ax = axa)$. (See [**27**].)

We introduce first some notation.

As in the previous section, $[n, n+m], n \geq 0, m \geq 1$, will denote the set $\{n, n+1, \ldots, n+m\}$.

Let $l \geq 1$ and let

$$C_{2l} = B_{2l+1} \quad , \quad C_{2l-1} = B'_{2l+1} \tag{7.2.1}$$

As mentioned in 7.1.3, C_n generates $\mathbf{VM}(R_{n+1} = S_{n+1})$, for $n \geq 1$.

For each subset I of $[0, l]$ containing 0, let $\mathrm{IM}(A, I)$ be the set of all maps $\triangle : I \to \mathcal{P}(A)$ such that

$$i < j \Longrightarrow \triangle(i) \subset \triangle(j) \qquad (i, j \in I),$$

where the inclusion above is strict.

For each $I \subseteq [0, l]$, $I = \{i_0, i_1, \ldots, i_j\}$ where $i_0 < i_1 < \cdots < i_j$ and $i_0 = 0$, each $\triangle \in \mathrm{IM}(A, I)$ and each $\{0\} \subseteq J \subseteq I$, let $\mathrm{INT}(A, J, \triangle)$ be the set of all maps

$\lambda : J \setminus \{0\} \to A$ such that

$$\lambda(i_k) \in \triangle(i_k) \setminus \triangle(i_{k-1}) \qquad (i_k \in J). \tag{7.2.2}$$

Notice that i_{k-1} may be outside J.

For $I \subseteq [0, l]$ and $\triangle \in \mathrm{IM}(A, I)$, we denote by $\triangle^*(i)$ the language $(\triangle(i))^*$.

Let $\{0\} \subseteq J \subseteq I \subseteq [0, l]$, $\triangle \in \mathrm{IM}(A, I)$ and $\lambda \in \mathrm{INT}(A, J, \triangle)$. Suppose that $J \setminus \{0\} = \{p_1, \ldots, p_t\} \cup \{q_1, \ldots, q_s\}$, where $p_1, \ldots, p_t$ are even and $q_1, \ldots, q_s$ are odd, $p_1 < \cdots < p_t$, $q_1 < \cdots < q_s$. Define

$$L(J, \triangle, \lambda) = \triangle^*(q_s)\lambda(q_s) \ldots \triangle^*(q_1)\lambda(q_1)\triangle^*(0)\lambda(p_1)\triangle^*(p_1) \ldots \lambda(p_t)\triangle^*(p_t).$$

Example 7.2.3. Let $A = \{a, b, c, d, e, f\}$, $l = 4$, $I = \{0, 1, 2, 4\}$ and $J = \{0, 1, 4\}$. Let $\triangle : I \to \mathcal{P}(A)$ be defined by

$$\begin{aligned} \triangle(0) &= \{a\} \\ \triangle(1) &= \{a, b\} \\ \triangle(2) &= \{a, b, e\} \\ \triangle(4) &= \{a, b, d, e, f\} \end{aligned}$$

For $\lambda_1 : \{1, 4\} \to A$ defined by $\lambda_1(1) = b, \lambda_1(4) = d$ we have

$$\begin{aligned} L(J, \triangle, \lambda_1) &= \triangle^*(1)\lambda_1(1)\triangle^*(0)\lambda_1(4)\triangle^*(4) \\ &= \{a, b\}^* b\{a\}^* d\{a, b, d, e, f\}^*. \end{aligned}$$

For $\lambda_2 : \{1, 4\} \to A$ defined by $\lambda_2(1) = b, \lambda_2(4) = f$ we have

$$\begin{aligned} L(J, \triangle, \lambda_2) &= \triangle^*(1)\lambda_2(1)\triangle^*(0)\lambda_2(4)\triangle^*(4) \\ &= \{a, b\}^* b\{a\}^* f\{a, b, d, e, f\}^*. \end{aligned}$$

Definition 7.2.4. A language L is of *type* L_m $(m \geq 1)$, if there is $\triangle \in \mathrm{IM}(A, [0, m])$ and $\lambda \in \mathrm{INT}(A, [0, m], \triangle)$ such that $L = L([0, m], \triangle, \lambda)$. A language is of *type* $\overline{L}_m$ $(m \geq 1)$, if $\overline{L}$ is of type L_m.

Notice that if m is even the languages of type L_m are the ones of the form

$$\triangle^*(m-1)\lambda(m-1)\ldots\triangle^*(1)\lambda(1)\triangle^*(0)\lambda(2)\triangle^*(2)\ldots\lambda(m)\triangle^*(m)$$

and if m is odd, are the ones of the form

$$\triangle^*(m)\lambda(m)\ldots\triangle^*(1)\lambda(1)\triangle^*(0)\lambda(2)\triangle^*(2)\ldots\lambda(m-1)\triangle^*(m-1).$$

Example 7.2.5. Let $A = \{a, b, c, d, e, f\}$, $l = 4$, and let $\triangle : [0, 4] \to \mathcal{P}(A)$ be defined by

$$\triangle(0) = \{a\}$$
$$\triangle(1) = \{a, b\}$$
$$\triangle(2) = \{a, b, e\}$$
$$\triangle(3) = \{a, b, c, e\}$$
$$\triangle(4) = \{a, b, c, e, f\}$$

There is only one map $\lambda : [1, 4] \to A$ satisfying (7.2.2), namely the map defined by $\lambda(1) = b, \lambda(2) = e, \lambda(3) = c$ and $\lambda(4) = f$. The language

$$\begin{aligned} L([0,4], \triangle, \lambda) &= \triangle^*(3)\lambda(3)\triangle^*(1)\lambda(1)\triangle^*(0)\lambda(2)\triangle^*(2)\lambda(4)\triangle^*(4) \\ &= \{a, b, c, e\}^* c\{a, b\}^* b\{a\}^* e\{a, b, e\}^* f\{a, b, c, e, f\}^*. \end{aligned}$$

is of type L_4.

We now state the theorem. We notice that because of Eilenberg's Theorem it is enough to describe the varieties of languages corresponding to the irreducible varieties of **LBM**.

Theorem 7.2.6. *Let $n \geq 1$. For each alphabet A, $A^*(R_{n+1} = S_{n+1})$ is the boolean algebra generated by the languages of type L_p, $1 \leq p \leq n$.*

Moreover, if $|A| = N < n$,

$$A^*(\mathcal{FBM}) = A^*(R_{n+1} = S_{n+1}) = A^*(R_{N+1} = S_{N+1}).$$

Before proving this we show that the result above cannot be improved.

Remark 7.2.7. Let $A = \{a, b\}$. By 7.2.6, $A^*(R_3 = S_3) = A^*(\mathcal{FBM})$. Thus, by 6.2.3, we get

$$|A^*(R_3 = S_3)| = 2^{|F_A(x^2=x)|} = 2^7. \quad \text{(See 3.1.5 and 3.1.1.)}$$

Now, the languages of type L_2 are the ones of the form

$$\triangle^*(1)\lambda(1)\triangle^*(0)\lambda(2)\triangle^*(2)$$

where

$$\triangle(0) \subset \triangle(1) \subset \triangle(2) \subseteq \{a, b\},$$

$$\lambda(1) \in \triangle(1) \setminus \triangle(0)$$

and

$$\lambda(2) \in \triangle(2) \setminus \triangle(1).$$

Therefore, the languages of type L_2 are

$$L = a^*abA^*$$

and

$$L' = b^*baA^*.$$

Since $L \cap L' = \emptyset$ and $A^* \setminus (L \cup L') = \{a^*\} \cup \{b^*\} \neq \emptyset$, the languages$L$, L' and $L'' = A^* \setminus (L \cup L')$ form a partition of A^*. Therefore, the boolean algebra generated by L, L' and L'', which is still the boolean algebra generated by L and L', has cardinality 2^3. Hence, L and L' do not generate $A^*(R_3 = S_3)$.

Our main theorem follows immediately from a series of lemmas which we now state. The proofs of the lemmas are deferred for the moment.

Lemma 7.2.8. *Let $n \geq 1$ and let $f : A \to C_n$. Let* id *be the identity in C_n. Define $\triangle : [0, n] \to \mathcal{P}(A)$ by $\triangle(0) = f^{-1}(\mathrm{id})$, $\triangle(i) = f^{-1}(C_i)$, $1 \leq i \leq n$, and let*

$$J = \{i \in [1, n] \ : \ \triangle(i) \setminus \triangle(i-1) \neq \emptyset\} \cup \{0\}.$$

Then A^* is a union of the sets $L(I, \triangle, \lambda)$, where $\{0\} \subseteq I \subseteq J$ and $\lambda \in \mathrm{INT}(A, I, \triangle)$.

Lemma 7.2.9. *Let $n \geq 1$. For each alphabet A, $A^*(R_{n+1} = S_{n+1})$ is contained in the boolean algebra generated by the sets $L(I, \triangle, \lambda)$, where $\{0\} \subseteq I \subseteq [0, n]$, $\triangle \in \mathrm{IM}(A, I)$ and $\lambda \in \mathrm{INT}(A, I, \triangle)$.*

Lemma 7.2.10. *Let $n \geq 1$. For each alphabet A, $A^*(R_{n+1} = S_{n+1})$ is contained in the boolean algebra generated by the languages of type L_p, $0 \leq p \leq n$ and $\overline{L}_p$, $0 \leq p \leq n-1$.*

Lemma 7.2.11. *Let $n \geq 1$. For each alphabet A, $A^*(R_{n+1} = S_{n+1})$ is contained in the boolean algebra generated by the languages of type L_p, $1 \leq p \leq n$.*

Lemma 7.2.12. *Let A be an alphabet and let $1 \leq p \leq n$. If L is a language of type L_p, then $L \in A^*(R_{n+1} = S_{n+1})$.*

It is now clear how these lemmas are used to prove Theorem 7.2.6.

Proof of Theorem 7.2.6. The first part of the statement follows immediately from Lemmas 7.2.11 and 7.2.12.

Now, if $|A| = N < n$, from 6.2.5 we get

$$A^*(R_{n+1} = S_{n+1}) = A^*(R_{N+1} = S_{N+1}).$$

It is also clear that $A^*(\mathcal{FBM}) = A^*(R_{N+1} = S_{N+1})$, since

$$\mathbf{FBM} = \bigvee\{\mathbf{VM}(R_n = S_n) \ : \ n \geq 2\}.$$

We now prove Lemmas 7.2.8 to 7.2.12.

Proof of Lemma 7.2.8. The proof is by induction on $n \geq 1$. For $n = 1$, $C_1 = B_3'$. If $x_1 \ldots x_r \in A^+$, then either $x_1, \ldots, x_r \in f^{-1}(\mathrm{id})$ or there is $i \in [1, r]$ such that $x_i \notin f^{-1}(\mathrm{id})$. In the first case

$$x_1 \ldots x_r \in (f^{-1}(\mathrm{id}))^* = \triangle^*(0) = L(\{0\}, \triangle, \lambda),$$

where $\lambda = \emptyset$.

In the second case let i_0 be the biggest $i \in [1, r]$ such that $x_i \notin f^{-1}(\mathrm{id})$. Then $1 \in J$, $x_{i_0} \in A \setminus \triangle(0) = \triangle(1) \setminus \triangle(0)$ and

$$x_1 \ldots x_r \in \triangle^*(1) x_{i_0} \triangle^*(0) = L(\{0, 1\}, \triangle, \lambda),$$

where $\lambda(1) = x_{i_0}$.

Let $n \geq 2$. There are two cases to consider: n even and n odd. If n is even , say $n = 2l$, $l \geq 1$, then, according to (7.2.1),

$$C_n = B_{2l+1} = Z_{2l+1} \cup B'_{2l+1} = Z_{2l+1} \cup C_{n-1}.$$

If $f^{-1}(C_{n-1}) \neq \emptyset$, let g be the restriction of f to $f^{-1}(C_{n-1})$, $\triangle' = \triangle|_{[0,n-1]}$ and

$$\begin{aligned} J' &= \{i \in [1, n-1] \ : \ \triangle'(i) \setminus \triangle'(i-1) \neq \emptyset\} \cup \{0\} \\ &= \{i \in [1, n-1] \ : \ \triangle(i) \setminus \triangle(i-1) \neq \emptyset\} \cup \{0\}. \end{aligned}$$

Let $x_1 \dots x_r \in A^+$. Either $x_1, \dots, x_r \in f^{-1}(C_{n-1})$ or there is $j \in [1, r]$ such that $x_j \notin f^{-1}(C_{n-1})$. In the first case, induction on n implies that there exist I', $\{0\} \subseteq I' \subseteq J' \subseteq J$, and $\lambda' \in \mathrm{INT}(A, I', \triangle') = \mathrm{INT}(A, I', \triangle)$ such that

$$x_1 \dots x_r \in L(I', \triangle', \lambda') = L(I', \triangle, \lambda').$$

In the second case, let j_0 be the least $j \in [1, r]$ such that $x_j \notin f^{-1}(C_{n-1})$. Then

$$x_{j_0+1} \dots x_r \in \left(f^{-1}(C_n)\right)^* = \triangle^*(n).$$

If $j_0 = 1$

$$x_1 \dots x_r \in \triangle^*(0) x_1 \triangle^*(n) = L(\{0, n\}, \triangle, \lambda),$$

where $\lambda(n) = x_1 \in \triangle(n) \setminus \triangle(n-1)$. If $j_0 > 1$, then $x_1, \dots, x_{j_0-1} \in f^{-1}(C_{n-1})$ and by the induction hypothesis there exist $I'' \subseteq J'$ and $\lambda'' \in \mathrm{INT}(A, I'', \triangle')$ such that

$$x_1 \dots x_{j_0-1} \in L(I'', \triangle', \lambda'') = L(I'', \triangle, \lambda'').$$

Therefore

$$x_1 \dots x_r \in L(I'', \triangle, \lambda'') x_{j_0} \triangle^*(n) = L(I'' \cup \{n\}, \triangle, \lambda''')$$

where $\lambda'''(n) = x_{j_0}$ and $\lambda'''(i) = \lambda''(i)$ for $i \in I''$.

Notice that if $f^{-1}(C_{n-1}) = \emptyset$, then $J = \{n\}$ and A^* is a union of the sets $\triangle^*(0) = L(\{0\}, \triangle, \emptyset)$ and $L(\{0, n\}, \triangle, \lambda)$, where $\lambda(n) \in A = \triangle(n) \setminus \triangle(n-1)$.

For n odd, say $n = 2l+1, l \geq 1$, we have $C_n = W_{2l+3} \cup C_{n-1}$ and the argument goes along the dual lines as the one for n even. (Recall 7.1.2 for the structure of the $B'_n s$.)

Proof of Lemma 7.2.9. Let A be an alphabet and $n \geq 1$. By 6.2.1, $A^*(R_{n+1} = S_{n+1})$ is the boolean algebra generated by the inverse images of the elements of C_n, by morphisms from A^* into C_n. Let f be such a morphism, determined by $f_0 : A \to C_n$ and let $y \in C_n$. It is enough to prove that $f^{-1}(y)$ is a union of the sets $L(I, \triangle, \lambda)$, where $I \subseteq [0, n]$, $\triangle \in \mathrm{IM}(A, I)$ and $\lambda \in \mathrm{INT}(A, I, \triangle)$, since this implies the containment of the two boolean algebras considered.

Let $x \in f^{-1}(y)$. By 7.2.8, x belongs to one of the sets defined there, namely, $L = L(I, \triangle, \lambda)$ where $\{0\} \subseteq I \subseteq J$ and $\lambda \in \mathrm{INT}(A, I, \triangle)$. It suffices to show that $|f(L)| = 1$, since this implies that $L \subseteq f^{-1}(y)$.

Let the set of odd numbers in I be $\{q_1, \ldots, q_s\}$ and the set of even numbers in I be $\{p_1, \ldots, p_t\}$, where $q_1 < \cdots < q_s$ and $p_1 < \cdots < p_t$. Then

$$L = \triangle^*(q_s)\lambda(q_s)\ldots\triangle^*(q_1)\lambda(q_1)\triangle^*(0)\lambda(p_1)\triangle^*(p_1)\ldots\lambda(p_t)\triangle^*(p_t)$$

and

$$f(L) \in B'_{q_s+2} f\big(\lambda(q_s)\big)\ldots B'_{q_1+2} f\big(\lambda(q_1)\big)\{\mathrm{id}\} f\big(\lambda(p_1)\big)B_{p_1+1}\ldots f\big(\lambda(p_t)\big)B_{p_t+1},$$

since

$$\triangle(q_l) = f_0^{-1}(C_{q_l}) = f_0^{-1}(B'_{q_l+2}) \qquad (1 \leq l \leq s)$$

and

$$\triangle(p_m) = f_0^{-1}(C_{p_m}) = f_0^{-1}(B_{p_m+1}) \qquad (1 \leq m \leq t).$$

But

$$f\big(\lambda(q_l)\big) \in B'_{q_l+2} \setminus B_{q_l} \qquad (1 \leq l \leq s)$$

and

$$f\big(\lambda(p_m)\big) \in B_{p_m+1} \setminus B'_{p_m+1} \qquad (1 \le m \le t),$$

since

$$\lambda(q_l) \in \triangle(q_l) \setminus \triangle(q_l - 1) = f_0^{-1}(B'_{q_l+2}) \setminus f_0^{-1}(B_{q_l}) \qquad (1 \le l \le s)$$

and

$$\lambda(p_m) \in \triangle(p_m) \setminus \triangle(p_m - 1) = f_0^{-1}(B_{p_m+1}) \setminus f_0^{-1}(B'_{p_m+1}) \qquad (1 \le m \le t).$$

Therefore, by 7.1.2, $f\big(\lambda(q_l)\big)$ is a right zero of B'_{q_l+2} and $f\big(\lambda(p_m)\big)$ is a left zero of B_{p_m+1}, for $1 \le l \le s$ and $1 \le m \le t$. Consequently,

$$f(L) = \{f\big(\lambda(q_s)\big) \dots f\big(\lambda(q_1)\big) f\big(\lambda(p_1)\big) \dots f\big(\lambda(p_t)\big)\}$$

has cardinality one, as required. Hence $L \subseteq f^{-1}(y)$.

Proof of Lemma 7.2.10. For each $I \subseteq [0, n]$ let

$$\delta(I) = \big| \{i \,:\, 2 \le i \le \max I \,,\, i \notin I\} \big|.$$

Notice first that the languages $L(I, \triangle, \lambda)$ such that $0 \in I$, $\triangle \in \mathrm{IM}(A, I)$, $\lambda \in \mathrm{INT}(A, I, \triangle)$ and $\delta(I) = 0$ are of type L_p $(0 \le p \le n)$ or $\overline{L}_p (0 \le p \le n-1)$. Indeed, if $L = L(I, \triangle, \lambda)$ is such a language and $1 \in I$, then L is of type $L_{\max I}$. If $1 \notin I$, define $\chi : [0, n-1] \to [0, n]$ by

$$\chi(i) = \begin{cases} i, & \text{if } i = 0; \\ i + 1, & \text{if } i > 0. \end{cases}$$

and let $I' = \chi^{-1}(I)$, $\chi' = \chi|_{I'}$, $\chi'' = \chi|_{I' \setminus \{0\}}$, $\lambda' = \lambda \circ \chi'$ and $\triangle' = \triangle \circ \chi'$. Then $\delta(I') = 0$ and $0, 1 \in I'$. Thus, $\overline{L} = L(I', \triangle', \lambda')$ is a language of type $L_{\max I'}$.

We prove that $A^*(R_{n+1} = S_{n+1})$ is contained in the boolean algebra generated by the languages $L(I,\triangle,\lambda)$ such that $\{0\} \subseteq I \subseteq [0,n]$, $\triangle \in \mathrm{IM}(A,I)$, $\lambda \in \mathrm{INT}(A,I,\triangle)$ and $\delta(I) = 0$. In order to do this we show that for all $k \geq 1$, the boolean algebra generated by the $L(I,\triangle,\lambda)$ such that $\{0\} \subseteq I \subseteq [0,n]$, $\triangle \in \mathrm{IM}(A,I)$, $\lambda \in \mathrm{INT}(A,I,\triangle)$ and $\delta(I) \leq k$ is contained in the boolean algebra generated by those $L = L(I,\triangle,\lambda)$ such that $\delta(I) \leq k-1$. The essential feature of the argument will be an application of cancellation and equidivisibility. (See 2.1.11).

Notice that if $I \subseteq [0,n]$, $\delta(I) \leq n-2$.

Let $L = L(I,\triangle,\lambda)$ be such that $\delta(I) \leq k$ and let i_0 be the least $i \in [2,n]$ such that $i \notin I$. Then, $i_0 < \max I$. There are two cases to consider : $i_0 + 1 \notin I$ and $i_0 + 1 \in I$.

If $i_0 + 1 \notin I$, define $\chi : [0,n-2] \to [0,n]$ by

$$\chi(i) = \begin{cases} i, & \text{if } i \leq i_0 - 1; \\ i+2, & \text{otherwise.} \end{cases}$$

Let $I' = \chi^{-1}(I)$, $\chi' = \chi|_{I'}, \chi'' = \chi|_{I'\setminus\{0\}}$, $\lambda' = \lambda \circ \chi'$ and $\triangle' = \triangle \circ \chi'$. Then $\delta(I') \leq k-2$ and $L(I,\triangle,\lambda) = L(I',\triangle',\lambda')$.

Suppose now that $i_0 + 1 \in I$. If $i_0 = 2$ and $1 \notin I$, that is, if $1,2 \notin I$, the reasoning made in the case $i_0, i_0 + 1 \notin I$ can be used. Finally, if $i_0 = 2$ and $1 \in I$ or $i_0 \neq 2$, the situation is : $i_0 - 1$, $i_0 + 1 \in I$, $i_0 \notin I$. In this case, let $I' = I \setminus \{i_0 + 1\}$, $\lambda' = \lambda|_{(I'\setminus\{0\})}$ and define $\triangle' : I' \to \mathcal{P}(A)$ by

$$\triangle'(i_0 - 1) = \triangle(i_0 + 1) \quad , \quad \triangle'(j) = \triangle(j) \quad (j \neq i_0 - 1).$$

Define $I'' = \{j \in J : j \geq i_0 + 1\} \cup \{0\}$, $\lambda'' = \lambda|_{(I''\setminus\{0\})}$ and $\triangle'' : I'' \to \mathcal{P}(A)$ by

$$\triangle''(0) = \triangle(i_0 - 1) \quad , \quad \triangle''(j) = \triangle(j) \quad (j \geq i_0 + 1).$$

Notice that $\delta(I'') \leq k-1$ and $i_0, i_0+1 \notin I'$.

We will see that

$$L(I,\triangle,\lambda) = L(I',\triangle',\lambda') \cap L(I'',t'',l''). \tag{7.2.10.1}$$

Let the set of odd numbers in I be $X = \{q_1,\ldots,q_s\}$ and the set of even numbers in I be $Y = \{p_1,\ldots,p_t\}$, where $q_1 < \cdots < q_s$ and $p_1 < \cdots < p_t$. Now, $i_0 - 1$ and $i_0 + 1$ have the same parity. Suppose that $i_0 - 1 = p_\mu$ and $i_0 + 1 = p_{\mu+1}$, for some $\mu \in [1,t]$. We have

$$L(I,\triangle,\lambda) = \triangle^*(q_s)\lambda(q_s)\ldots\triangle^*(q_1)\lambda(q_1)\triangle^*(0)\lambda(p_1)\triangle^*(p_1)\ldots\lambda(p_t)\triangle^*(p_t)$$

and

$$\begin{aligned} L(I',\triangle',\lambda') = {} & \triangle^*(q_s)\lambda(q_s)\ldots\triangle^*(q_1)\lambda(q_1)\triangle^*(0)\lambda(p_1)\triangle^*(p_1)\ldots \\ & \ldots\lambda(p_{\mu-1})\triangle^*(p_{\mu-1})\lambda(p_\mu)\triangle^*(p_{\mu+1})\ldots\lambda(p_t)\triangle^*(p_t). \end{aligned}$$

If $X \cap \{j \in I : j > i_0 + 1\} = \emptyset$, we have

$$L(I'',\triangle'',\lambda'') = \triangle^*(p_\mu)\lambda(p_{\mu+1})\triangle^*(p_{\mu+1})\ldots\lambda(p_t)\triangle^*(p_t) \tag{7.2.10.2}$$

and if $X \cap \{j \in I : j > i_0 + 1\} = \{q_\sigma,\ldots,q_s\}$, we have

$$\begin{aligned} L(I'',\triangle'',\lambda'') = {} & \triangle^*(q_s)\lambda(q_s)\ldots\triangle^*(q_\sigma)\lambda(q_\sigma)\triangle^*(p_\mu)\lambda(p_{\mu+1})\triangle^*(p_{\mu+1})\ldots \\ & \ldots\lambda(p_t)\triangle^*(p_t). \end{aligned} \tag{7.2.10.3}$$

Clearly,

$$L(I,\triangle,\lambda) \subseteq L(I',\triangle',\lambda') \cap L(I'',\triangle'',\lambda'').$$

Now, let $u \in L(I',\triangle',\lambda') \cap L(I'',\triangle'',\lambda'')$. Since $u \in L(I',\triangle',\lambda')$, we have

$$u = u_{q_s}\lambda(q_s)\ldots u_{q_1}\lambda(q_1)u_0\lambda(p_1)u_{p_1}\ldots\lambda(p_{\mu-1})u_{p_{\mu-1}}\lambda(p_\mu)u_{p_{\mu+1}}\ldots\lambda(p_t)u_{p_t},$$

where $u_{q_i} \in \triangle^*(q_i)$, $u_{p_j} \in \triangle^*(p_j)$, for $1 \le i \le s$, $j \in \{1, \ldots, \mu - 1, \mu + 1, \ldots, t\}$.

On the other hand, since $u \in L(I'', \triangle'', \lambda'')$, we have

$$u = u'_{p_\mu} \lambda(p_{\mu+1}) u'_{p_{\mu+1}} \ldots \lambda(p_t) u'_{p_t}$$

if (7.2.10.2) occurs and

$$u = u''_{q_s} \lambda(q_s) \ldots u''_{q_\sigma} \lambda(q_\sigma) u''_{p_\mu} \lambda(p_{\mu+1}) u''_{p_{\mu+1}} \ldots \lambda(p_t) u''_{p_t}$$

if (7.2.10.3) occurs, where $u'_{p_j}, u''_{p_j} \in \triangle^*(p_j)$ and $u''_{q_l} \in \triangle^*(q_l)$, for $j \in \{\mu, \mu + 1, \ldots, t\}$ and $\lambda \in \{\sigma, \sigma + 1, \ldots, s\}$.

In the first case, since

$$\triangle(q_1), \ldots, \triangle(q_s), \triangle(p_1), \ldots, \triangle(p_{\mu-1}) \subseteq \triangle(p_\mu) \subseteq \triangle(p_{\mu+1}) \subseteq \cdots \subseteq \triangle(p_t),$$

cancellation on the right yields

$$u_{p_t} = u'_{p_t}, \ldots, u_{p_{\mu+2}} = u'_{p_{\mu+2}}$$

and

$$u_{q_s} \lambda(q_s) \ldots u_{q_1} \lambda(q_1) u_0 \lambda(p_1) u_{p_1} \ldots \lambda(p_{\mu-1}) u_{p_{\mu-1}} \lambda(p_\mu) u_{p_{\mu+1}} = u'_{p_\mu} \lambda(p_{\mu+1}) u'_{p_{\mu+1}}.$$

Again, since

$$u_{q_s} \lambda(q_s) \ldots u_{q_1} \lambda(q_1) u_0 \lambda(p_1) u_{p_1} \ldots \lambda(p_{\mu-1}) u_{p_{\mu-1}} \lambda(p_\mu) \in \triangle^*(p_\mu),$$

we deduce that

$$u_{p_{\mu+1}} = x \lambda(p_{\mu+1}) u'_{p_{\mu+1}},$$

where x is a right factor of u'_{p_μ} (possibly empty). Hence

$$u = u_{q_s} \lambda(q_s) \ldots u_{q_1} \lambda(q_1) u_0 \lambda(p_1) u_{p_1} \ldots$$
$$\ldots \lambda(p_{\mu-1}) u_{p_{\mu-1}} \lambda(p_\mu) x \lambda(p_{\mu+1}) u'_{p_{\mu+1}} \lambda(p_{\mu+2}) u_{p_{\mu+2}} \ldots \lambda(p_t) u_{p_t} \in L.$$

In the second case, a similar reasoning gives that $u \in L$.

We have shown (7.2.10.1) when $i_0 - 1, i_0 + 1$ are even. If $i_0 - 1, i_0 + 1$ are odd the proof is similar.

We have now proved the containment of the two boolean algebras, therefore, we deduce that $A^*(R_{n+1} = S_{n+1})$ is indeed contained in the boolean algebra generated by the $L(I, \triangle, \lambda)$ such that $\{0\} \subseteq I \subseteq [0, n]$, $\triangle \in \mathrm{IM}(A, I)$, $\lambda \in \mathrm{INT}(A, I, \triangle)$ and $\delta(I) = 0$, as required.

Proof of Lemma 7.2.11. By 7.2.10, $A^*(R_{n+1} = S_{n+1})$ is contained in the boolean algebra generated by the languages of type L_p, $0 \leq p \leq n$ and $\overline{L}_p$, $0 \leq p \leq n-1$. We will see that if L is a language of type $\overline{L}_p$, $1 \leq p \leq n-1$, then L is in the boolean algebra generated by the languages of type L_p, $1 \leq p \leq n$. As in the previous proof, the conclusions will be achieved using cancellation and equidivisibility and we omit the explicit justifications.

Let $p \geq 1$ and $L = L([0,p], \triangle, \lambda)$, where $\triangle \in \mathrm{IM}(A, [0,p])$ and $\lambda \in \mathrm{INT}(A, [0,p], \triangle)$, be a language of type $\overline{L}_p$.

Suppose first that $\triangle(0) = \emptyset$, $|\triangle(1)| = 1$.

Suppose now that p is odd. For $p = 1$, we get

$$L = \lambda(1)\triangle^*(1) = \lambda(1)(\lambda(1))^* = (\lambda(1))^*\lambda(1)\triangle^*(0).$$

Thus, L is a language of type L_1.

For $p > 1$, we have

$$\begin{aligned} L &= \triangle^*(p-1)\lambda(p-1)\ldots\triangle^*(2)\lambda(2)\lambda(1)(\lambda(1))^*\lambda(3)\triangle^*(3)\ldots\lambda(p)\triangle^*(p) \\ &= \triangle^*(p-1)\lambda(p-1)\ldots\triangle^*(2)\lambda(2)(\lambda(1))^*\lambda(1)\lambda(3)\triangle^*(3)\ldots\lambda(p)\triangle^*(p). \end{aligned}$$

Using the mentioned cancellation and equidivisibility arguments we deduce that

$$L = L' \cap L'',$$

where

$$\begin{aligned} L' &= \triangle^*(p-1)\lambda(p-1)\ldots\triangle^*(2)\lambda(1)\lambda(3)\triangle^*(3)\ldots\lambda(p)\triangle^*(p) \\ &= \triangle^*(p-1)\lambda(p-1)\ldots\triangle^*(2)\lambda(1)\triangle^*(0)\lambda(3)\triangle^*(3)\ldots\lambda(p)\triangle^*(p) \end{aligned}$$

and

$$L'' = \triangle^*(p-1)\lambda(p-1)\ldots\triangle^*(2)\lambda(2)(\lambda(1))^*\lambda(3)\triangle^*(3)\ldots\lambda(p)\triangle^*(p).$$

Define $\triangle' : [0,p-1] \to \mathcal{P}(A)$ by

$$\triangle'(i) = \begin{cases} \triangle(i), & \text{if } i = 0; \\ \triangle(i+1), & \text{if } i > 0 \end{cases}$$

and $\lambda' : [1,p-1] \to A$ by

$$\lambda'(i) = \begin{cases} \lambda(i), & \text{if } i = 1; \\ \lambda(i+1), & \text{if } i > 1. \end{cases}$$

Then $L' = L([0,p-1],\triangle',\lambda')$ is a language of type L_{p-1}.

Define also $\chi : [0,p-1] \to [0,p]$ by $\chi(i) = i+1$ and let $\triangle'' = \triangle \circ \chi$, $\lambda'' = \lambda \circ \chi|_{[1,p-1]}$. Then $L'' = L([0,p-1],\triangle'',\lambda'')$ is a language of type L_{p-1}.

For p even we get

$$L = \triangle^*(p)\lambda(p)\ldots\triangle^*(2)\lambda(2)\lambda(1)(\lambda(1))^*\ldots\lambda(p-1)\triangle^*(p-1)$$

and a similar argument yields again that $L = L' \cap L''$, where L' and L'' are languages of type L_{p-1}. Therefore, L is in the boolean algebra generated by the languages of type L_p, $0 \le p \le n$.

We now prove the statement under the weaker assumption $\triangle(0) = \emptyset$.

Again, suppose first that p is odd. For $p = 1$ we get

$$\begin{aligned} L &= \lambda(1)\triangle^*(1) \\ &= L' \cup \bigcup_{a \in \triangle(1)\setminus\{\lambda(1)\}} L_a, \end{aligned}$$

where $L' = \lambda(1)(\lambda(1))^*$ and

$$L_a = (\lambda(1))^*\lambda(1)a\triangle^*(1) = (\lambda(1))^*\lambda(1)\triangle^*(0)a\triangle^*(1),$$

for each $a \in \triangle(1) \setminus \{\lambda(1)\}$.

For $p > 1$ we get

$$\begin{aligned} L &= \triangle^*(p-1)\lambda(p-1)\ldots\triangle^*(2)\lambda(2)\lambda(1)\triangle^*(1)\lambda(3)\triangle^*(3)\ldots\lambda(p)\triangle^*(p) \\ &= L' \cup \bigcup_{a\in\triangle(1)\setminus\{\lambda(1)\}} L_a, \end{aligned}$$

where

$$L' = \triangle^*(p-1)\lambda(p-1)\ldots\triangle^*(2)\lambda(2)\lambda(1)(\lambda(1))^*\lambda(3)\triangle^*(3)\ldots\lambda(p)\triangle^*(p)$$

and

$$\begin{aligned} L_a &= \triangle^*(p-1)\ldots\triangle^*(2)\lambda(2)(\lambda(1))^*\lambda(1)a\triangle^*(1)\lambda(3)\triangle^*(3)\ldots\lambda(p)\triangle^*(p) \\ &= \triangle^*(p-1)\ldots\triangle^*(2)\lambda(2)(\lambda(1))^*\lambda(1)\emptyset^*a\triangle^*(1)\lambda(3)\triangle^*(3)\ldots\lambda(p)\triangle^*(p), \end{aligned}$$

for each $a \in \triangle(1) \setminus \{\lambda(1)\}$.

In both cases, the language L' satisfies the assumption $\triangle(0) = \emptyset$, $|\triangle(1)| = 1$. By above, L' is in the boolean algebra generated by the languages of type L_p, $0 \leq p \leq n$.

Now let $a \in \triangle(1) \setminus \{\lambda(1)\}$ and define $\triangle'' : [0, p+1] \to \mathcal{P}(A)$ by

$$\begin{aligned} \triangle''(0) &= \triangle(0) \\ \triangle''(1) &= \{\lambda(1)\} \\ \triangle''(i+1) &= \triangle(i)\ ,\ i \geq 1. \end{aligned}$$

Define also $\lambda'' : [1, p+1] \to A$ by

$$\begin{aligned} \lambda''(1) &= \lambda(1) \\ \lambda''(2) &= a \\ \lambda''(i+1) &= \lambda(i)\ ,\ i \geq 2. \end{aligned}$$

In both cases, $L_a = L([0, p+1], \Delta'', \lambda'')$ is a language of type L_{p+1}. Thus, L is in the required boolean algebra.

For p even the reasoning is similar.

Finally, we prove the statement without any additional assumption. Again, we assume p odd, the case of p even being similar.

For $p = 1$ we have

$$\begin{aligned} L &= \Delta^*(0)\lambda(1)\Delta^*(1) \\ &= L' \cup \bigcup_{a \in \Delta(0)} L_a, \end{aligned}$$

where $L' = \lambda(1)\Delta^*(1)$ and for each $a \in \Delta(0)$,

$$L_a = \Delta^*(0)a\lambda(1)\Delta^*(1) = \Delta^*(0)a\emptyset^*\lambda(1)\Delta^*(1).$$

For $p > 1$ we have

$$\begin{aligned} L &= \Delta^*(p-1)\lambda(p-1)\ldots\Delta^*(2)\lambda(2)\Delta^*(0)\lambda(1)\Delta^*(1)\ldots\lambda(p)\Delta^*(p) \\ &= L' \cup \bigcup_{a \in \Delta(0)} L_a, \end{aligned}$$

where

$$L' = \Delta^*(p-1)\lambda(p-1)\ldots\Delta^*(2)\lambda(2)\lambda(1)\Delta^*(1)\ldots\lambda(p)\Delta^*(p)$$

and for each $a \in \Delta(0)$

$$L_a = \Delta^*(p-1)\lambda(p-1)\ldots\Delta^*(2)\lambda(2)\Delta^*(0)a\lambda(1)\Delta^*(1)\ldots\lambda(p)\Delta^*(p).$$

In both cases , the language L' satisfies the assumption $\Delta(0) = \emptyset$. Hence, by above, L' is in the boolean algebra generated by the languages of type L_p, $0 \le p \le n$.

Now let $a \in \Delta(0)$ and define $\Delta' : [0, p+1] \to \mathcal{P}(A)$ by

$$\begin{aligned} &\Delta'(0) = \emptyset \\ &\Delta'(i+1) = \Delta(i)\ , \ i \ge 0. \end{aligned}$$

Define also $\lambda' : [1, p+1] \to A$ by

$$\lambda'(1) = a$$

$$\lambda'(i+1) = \lambda(i)\ ,\ i \geq 1 .$$

In both cases, $L_a = L([0, p+1], \triangle', \lambda')$ is a language of type L_{p+1}. Therefore, L is in the boolean algebra generated by the languages of type L_p, $1 \leq p \leq n$.

Finally, we notice that if L is a language of type L_0 (or $\overline{L}_0$), then L is also in the boolean algebra required. (See [**20**, page 40].)

Proof of Lemma 7.2.12. The proof is by induction on $n \geq 1$. For $n = 1$ we get $L = \triangle^*(1)\lambda(1)\triangle^*(0)$, where $\triangle \in \mathrm{IM}(A, [0,1])$ and $\lambda \in \mathrm{INT}(A, [0,1], \triangle)$. Define $f_0 : A \to C_1 = B_3'$ by

$$f_0(x) = \begin{cases} b_2^3, & \text{if } x = \lambda(1); \\ b_1^3, & \text{if } x \in A \setminus \big(\triangle(0) \cup \{\lambda(1)\}\big); \\ \mathrm{id}, & \text{if } x \in \triangle(0). \end{cases}$$

We will show that $L = \triangle^*(1) \cap f^{-1}(b_2^3)$, where f is the extension of f_0 to A^*. Indeed, if $u \in L$, $u = u_1\lambda(1)u_0$, where $u_i \in \triangle^*(i)$, $0 \leq i \leq 1$. Hence

$$f(u) = f(u_1)b_2^3 f(u_0) = f(u_1)b_2^3 = b_2^3,$$

since, by 7.1.2, the elements of W_3 are right zeros of B_3'.

Conversely, let $u = u_1 \ldots u_k$, where $u_1, \ldots, u_k \in \triangle(1)$, be an element of $\triangle^*(1)$ such that $f(u) = b_2^3$. Let i_0 be the biggest $i \in [1, k]$ such that $u_i \notin \triangle(0)$. Then

$$\begin{aligned} f(u) &= f(u_1) \ldots f(u_{i_0}) && \text{since } f(\triangle(0)) = \{id\} \\ &= f(u_{i_0}) && \text{by 7.1.2(iii)} \\ &= b_\gamma^3 && \text{for some } \gamma \in \{1, 2\}. \end{aligned}$$

On the other hand, $f(u) = b_2^3$. Consequently, $\gamma = 2$ and $u_{i_0} = \lambda(1)$. Therefore, $u \in \triangle^*(1)\lambda(1)\triangle^*(0)$. Hence $L = \triangle^*(1) \cap f^{-1}(b_2^3) \in A^*(R_2 = S_2)$, as required.

Now, let $n \geq 2$ and let $L = L([0,p], \triangle, \lambda)$ be a language of type L_p, where $p > 1$. Let

$$2s + 1 = \max\big([0,p] \cap \{\text{ odd numbers}\}\big)$$

and

$$2t = \max\big([0,p] \cap \{\text{ even numbers}\}\big).$$

Then $s \geq 0$ and $t \geq 1$. Define $f_0 : A \to C_n$ by

$$f_0(x) = \begin{cases} b_2^3, & \text{if } x = \lambda(1)\ ; \\ b_1^3, & \text{if } x \in \triangle(1) \setminus \big(\triangle(0) \cup \{\lambda(1)\}\big); \\ \mathrm{id}, & \text{if } x \in \triangle(0); \\ a_2^3, & \text{if } x \in \triangle(2) \setminus \triangle(1); \\ b_{2i+1}^{2i+3}, & \text{if } x \in \triangle(2i+1) \setminus \triangle(2i) \quad \text{for } 0 < i \leq s \\ & \text{and, if } p \text{ is odd, for } x \in A \setminus \triangle(2s+1); \\ a_{2j-1}^{2j+1}, & \text{if } x \in \triangle(2j) \setminus \triangle(2j-1) \quad \text{for } 2 < j \leq t \\ & \text{and, if } p \text{ is even, for } x \in A \setminus \triangle(2t). \end{cases}$$

Let f be the extension of f_0 to A^* and define $I' = I \setminus \{1\}$, $\lambda' = \lambda|_{(I' \setminus \{0\})}$ and $\triangle' : I' \to \mathcal{P}(A)$ by

$$\triangle'(0) = \triangle(1) \quad , \quad \triangle'(i) = \triangle(i) \quad (i \neq 0).$$

Then $L' = L(I', \triangle', \lambda')$ is of type $\overline{L}_{p-1}$ and therefore $\overline{L'}$ is of type L_{p-1}. The induction hypothesis implies that $\overline{L'} \in A^*(R_n = S_n)$ and consequently

$$L' \in A^*(\overline{R}_n = \overline{S}_n) \subseteq A^*(R_{n+1} = S_{n+1}).$$

Let g be the restriction of f to L'. We will prove that $L = g^{-1}\left(b_{2s+2}^{2s+3} a_{2t}^{2t+1}\right)$. This will imply that $L \in A^*(R_{n+1} = S_{n+1})$, since

$$g^{-1}\left(b_{2s+2}^{2s+3} a_{2t}^{2t+1}\right) = f^{-1}\left(b_{2s+2}^{2s+3} a_{2t}^{2t+1}\right) \cap L'.$$

Let $u \in L$. Then

$$u = u_{2s+1}\lambda(2s+1)\ldots u_1\lambda(1)u_0\lambda(2)u_2\ldots\lambda(2t)u_{2t},$$

where $u_i \in \triangle^*(i), 1 \leq i \leq t, s \geq 0$ and $t \geq 1$.

If $s > 0, t > 1$, then

$$\begin{aligned} g(u) &= f(u_{2s+1})b_{2s+1}^{2s+3}\ldots f(u_3)b_3^5 f(u_1)b_2^3\{\mathrm{id}\}a_2^3 f(u_2)a_3^5 f(u_4)\ldots a_{2t-1}^{2t+1} f(u_{2t}) \\ &= b_{2s+1}^{2s+3}\ldots b_3^5 b_2^3 a_2^3 a_3^5 \ldots a_{2t-1}^{2t+1} \qquad \text{by 7.1.2} \\ &= b_{2s+1}^{2s+3} b_2^3 a_2^3 a_{2t-1}^{2t+1} \qquad \text{by 7.1.7} \\ &= b_{2s+2}^{2s+3} a_{2t}^{2t+1} \qquad \text{by 7.1.6.} \end{aligned}$$

If $s = 0, t > 1$, then

$$\begin{aligned} g(u) &= f(u_1)b_2^3\{id\}a_2^3 f(u_2)a_3^5 f(u_4)\ldots a_{2t-1}^{2t+1} f(u_{2t}) \\ &= b_2^3 a_2^3 a_3^5 \ldots a_{2t-1}^{2t+1} \qquad \text{by 7.1.2} \\ &= b_2^3 a_2^3 a_{2t-1}^{2t+1} \qquad \text{by 7.1.7} \\ &= b_2^3 a_{2t}^{2t+1} \qquad \text{by 7.1.6.} \end{aligned}$$

If $s > 0, t = 1$, then

$$\begin{aligned} g(u) &= f(u_{2s+1})b_{2s+1}^{2s+3}\ldots f(u_3)b_3^5 f(u_1)b_2^3\{id\}a_2^3 f(u_2) \\ &= b_{2s+1}^{2s+3}\ldots b_3^5 b_2^3 a_2^3 \qquad \text{by 7.1.2} \\ &= b_{2s+1}^{2s+3} b_2^3 a_2^3 \qquad \text{by 7.1.7} \\ &= b_{2s+2}^{2s+3} a_2^3 \qquad \text{by 7.1.6.} \end{aligned}$$

Finally, for $s = 0$, $t = 1$ we get

$$\begin{aligned} g(u) &= f(u_1) b_2^3 \{\mathrm{id}\} a_2^3 f(u_2) \\ &= b_2^3 a_2^3, \qquad \text{by 7.1.2.} \end{aligned}$$

Hence, in all cases,

$$g(u) = b_{2s+2}^{2s+3} a_{2t}^{2t+1}.$$

Conversely, let $u \in L'$ be such that $g(u) = b_{2s+2}^{2s+3} a_{2t}^{2t+1}$.

If $s > 0$, then

$$L' = \triangle^*(2s+1)\lambda(2s+1)\ldots\lambda(3)\triangle^*(1)\lambda(2)\triangle^*(2)\lambda(4)\ldots\lambda(2t)\triangle^*(2t)$$

and

$$u = u_{2s+1}\lambda(2s+1)\ldots\lambda(3)u_1\lambda(2)u_2\lambda(4)\ldots\lambda(2t)u_{2t},$$

where $u_i \in \triangle^*(i)$, $i \geq 1$.

Thus,

$$\begin{aligned} g(u) &= f(u_{2s+1}) b_{2s+1}^{2s+3} \ldots b_3^5 f(u_1) a_2^3 f(u_2) a_3^5 \ldots a_{2t-1}^{2t+1} f(u_{2t}), \\ &= b_{2s+1}^{2s+3} \ldots b_3^5 f(u_1) a_2^3 a_3^5 \ldots a_{2t-1}^{2t+1} \qquad \text{by 7.1.2} \\ &= b_{2s+1}^{2s+3} f(u_1) a_{2t}^{2t+1} \qquad \text{by 7.1.6 and 7.1.7.} \end{aligned}$$

If $s = 0$, then

$$L' = \triangle^*(1)\lambda(2)\triangle^*(2)\lambda(4)\ldots\lambda(2t)\triangle^*(2t).$$

and

$$u = u_1\lambda(2)u_2\lambda(4)\ldots\lambda(2t)u_{2t},$$

where $u_i \in \triangle^*(i)$, $i \geq 1$. Thus,

$$\begin{aligned} g(u) &= f(u_1) a_2^3 f(u_2) a_3^5 \ldots a_{2t-1}^{2t+1} f(u_{2t}) \\ &= f(u_1) a_2^3 a_3^5 \ldots a_{2t-1}^{2t+1} \qquad \text{by 7.1.2} \\ &= f(u_1) a_{2t}^{2t+1} \qquad \text{by 7.1.6 and 7.1.7.} \end{aligned}$$

We know that $f(u_1) \in B'_3$. On the other hand, a simple computation using 7.1.6(iii) yields

$$b_{2s+2}^{2s+3} a_{2t}^{2t+1} = \begin{cases} b_{2s+2-2t}^{2s+3}, & \text{if } t < s+1; \\ a_{2t-2s-2}^{2t+1}, & \text{if } t \geq s+1 \end{cases}$$

and

$$b_{2s+1}^{2s+3} a_{2t}^{2t+1} = \begin{cases} b_{2s+1-2t}^{2s+3} & \text{if } t < s+1; \\ a_{2t-2s-1}^{2t+1} & \text{if } t \geq s+1. \end{cases}$$

Therefore $b_{2s+2}^{2s+3} a_{2t}^{2t+1} \neq b_{2s+1}^{2s+3} a_{2t}^{2t+1}$ and thus $f(u_1) \neq \text{id}$. Since $f(u_1) \in B'_3$, we have that $f(u_1) = b_\gamma^3$ for some $\gamma \in \{1, 2\}$ and we get

$$\begin{aligned} g(u) &= \begin{cases} b_{2s+1}^{2s+3} b_\gamma^3 a_{2t}^{2t+1}, & \text{if } s > 0; \\ b_\gamma^3 a_{2t}^{2t+1}, & \text{if } s = 0, \end{cases} \\ &= \begin{cases} b_{2s+\gamma}^{2s+3} a_{2t}^{2t+1}, & \text{if } s > 0; \\ b_\gamma^3 a_{2t}^{2t+1}, & \text{if } s = 0, \end{cases} \qquad \text{by 7.1.6(ii).} \end{aligned}$$

Therefore $\gamma = 2$ and $f(u_1) = f(u_1^{i_0}) = b_2^3$, where $u_1^{i_0}$ is the last letter of u_1 such that $f(u_1^{i_0}) \neq \text{id}$. Hence $u_1 \in \Delta^*(1)\lambda(1)\Delta^*(0)$ and $u \in L$.

We finish this section with a sequence of examples that illustrate our procedure for proving Theorem 7.2.6. The next example illustrates 7.2.8.

Example 7.2.13. Let $A = \{a, b, c\}$ and let $n = 4$. Then $C_4 = B_5$. Let $f : A \to B_5$ be defined by

$$\begin{aligned} f(a) &= b_1^5 \\ f(b) &= b_2^5 \\ f(c) &= a_1^3. \end{aligned}$$

If $\triangle : [0,4] \to \mathcal{P}(A)$ is defined as in 7.2.8 we get

$$\triangle(0) = f^{-1}(\mathrm{id}) = \emptyset$$

$$\triangle(1) = f^{-1}(C_1) = f^{-1}(B_3') = \emptyset$$

$$\triangle(2) = f^{-1}(C_2) = f^{-1}(B_3) = \{c\}$$

$$\triangle(3) = f^{-1}(C_3) = f^{-1}(B_5') = \{a,b,c\} = A$$

$$\triangle(4) = f^{-1}(C_4) = f^{-1}(B_5) = \{a,b,c\} = A.$$

Also,

$$J = \{i \in [1,4] : \triangle(i) \setminus \triangle(i-1) \neq \emptyset\} \cup \{0\}$$
$$= \{0,2,3\}$$

For $I = \{0\}$, $\mathrm{INT}(A,I,\triangle) = \{\emptyset\}$ and $L(\{0\},\triangle,\emptyset) = \emptyset^*$.

For $I = \{0,2\}$, $\mathrm{INT}(A,I,\triangle) = \{\lambda\}$ where $\lambda : \{2\} \to A$ is defined by $\lambda(2) = c \in \triangle(2) \setminus \triangle(0)$. Hence

$$L(\{0,2\},\triangle,\lambda) = \triangle^*(0)\lambda(2)\triangle^*(2) = \emptyset^* c\{c\}^*.$$

For $I = \{0,3\}$, $\mathrm{INT}(A,I,\triangle) = \{\lambda_1,\lambda_2\}$ where$\lambda_1,\lambda_2 : \{3\} \to A$ are defined by $\lambda_1(3) = a \in \triangle(3) \setminus \triangle(2)$ and $\lambda_2(3) = b$. Hence

$$L(\{0,3\},\triangle,\lambda_1) = \triangle^*(3)\lambda_1(3)\triangle^*(0) = A^* a$$

and

$$L(\{0,3\},\triangle,\lambda_2) = \triangle^*(3)\lambda_2(3)\triangle^*(0) = A^* b.$$

Finally, for $I = \{0,2,3\}$, $\mathrm{INT}(A,I,\triangle) = \{\lambda_1,\lambda_2\}$ where $\lambda_1,\lambda_2 : \{2,3\} \to A$ are defined by

$$\lambda_1(2) = c \in \triangle(2) \setminus \triangle(0)$$

$$\lambda_1(3) = a \in \triangle(3) \setminus \triangle(2)$$

and

$$\lambda_2(2) = c \in \Delta(2) \setminus \Delta(0)$$

$$\lambda_2(3) = b \in \Delta(3) \setminus \Delta(2).$$

Hence

$$L(\{0,2,3\}, \Delta, \lambda_1) = \Delta^*(3)\lambda_1(3)\Delta^*(0)\lambda_1(2)\Delta^*(2) = A^*ac\{c\}^*$$

and

$$L(\{0,2,3\}, \Delta, \lambda_2) = \Delta^*(3)\lambda_2(3)\Delta^*(0)\lambda_2(2)\Delta^*(2) = A^*bc\{c\}^*.$$

All together we have obtained the languages

$$\{1\}, c\{c\}^*, A^*a, A^*b, A^*ac\{c\}^* \text{ and } A^*bc\{c\}^*.$$

It is obvious that A^* is a union of these languages. Indeed, if $1 \neq w \in A^*$ we have the following different possibilities:

(1) the last letter of w is a;

(2) the last letter of w is b;

(3) the last letter of w is c.

Then (1) corresponds to A^*a, (2) corresponds to A^*b and (3) corresponds to $c\{c\}^*$, $A^*ac\{c\}^*$ or $A^*bc\{c\}^*$, if w contains only c's, the last letter of w different from c is a or the last letter of w different from c is b, respectively.

The next example illustrates Lemmas 7.2.9–7.2.11.

Example 7.2.14. Let $n = 4$. Lemma 7.2.9 asserts that $A^*(R_5 = S_5)$ is contained in the boolean algebra generated by the sets of the kind

(1)A_0^*, $A_0 \subset A$;

(2)$A_1^* a_1 A_0^*$, $A_0 \subset A_1 \subseteq A, a_1 \in A_1 \setminus A_0$;

(3)$A_0^* a_2 A_2^*$, $A_0 \subset A_2 \subseteq A, a_2 \in A_2 \setminus A_0$;

(4)$A_1^* a_1 A_0^* a_2 A_2^*$, $A_0 \subset A_1 \subset A_2 \subseteq A, a_1 \in A_1 \setminus A_0, a_2 \in A_2 \setminus A_1$;

(5)$A_3^* a_3 A_0^*$, $A_0 \subset A_3 \subseteq A, a_3 \in A_3 \setminus A_0$;

(6)$A_3^* a_3 A_1^* a_1 A_0^*$, $A_0 \subset A_1 \subset A_3 \subseteq A, a_1 \in A_1 \setminus A_0, a_3 \in A_3 \setminus A_1$;

(7)$A_3^* a_3 A_0^* a_2 A_2^*$, $A_0 \subset A_2 \subset A_3 \subseteq A, a_2 \in A_2 \setminus A_0, a_3 \in A_3 \setminus A_2$;

(8)$A_3^* a_3 A_1^* a_1 A_0^* a_2 A_2^*$, $A_0 \subset A_1 \subset A_2 \subset A_3 \subseteq A, a_1 \in A_1 \setminus A_0, a_2 \in A_2 \setminus A_1, a_3 \in A_3 \setminus A_2$;

(9)$A_0^* a_4 A_4^*$, $A_0 \subset A_4 \subseteq A, a_4 \in A_4 \setminus A_0$;

(10)$A_1^* a_1 A_0^* a_4 A_4^*$, $A_0 \subset A_1 \subset A_4 \subseteq A, a_1 \in A_1 \setminus A_0, a_4 \in A_4 \setminus A_1$;

(11)$A_0^* a_2 A_2^* a_4 A_4^*$, $A_0 \subset A_2 \subset A_4 \subseteq A, a_2 \in A_2 \setminus A_0, a_4 \in A_4 \setminus A_2$;

(12)$A_1^* a_1 A_0^* a_2 A_2^* a_4 A_4^*$, $A_0 \subset A_1 \subset A_2 \subset A_4 \subseteq A, a_1 \in A_1 \setminus A_0, a_2 \in A_2 \setminus A_1, a_4 \in A_4 \setminus A_2$;

(13)$A_3^* a_3 A_0^* a_4 A_4^*$, $A_0 \subset A_3 \subset A_4 \subseteq A, a_3 \in A_3 \setminus A_0, a_4 \in A_4 \setminus A_3$;

(14)$A_3^* a_3 A_1^* a_1 A_0^* a_4 A_4^*$, $A_0 \subset A_1 \subset A_3 \subset A_4 \subseteq A, a_1 \in A_1 \setminus A_0, a_3 \in A_3 \setminus A_1, a_4 \in A_4 \setminus A_3$;

(15)$A_3^* a_3 A_0^* a_2 A_2^* a_4 A_4^*$, $A_0 \subset A_2 \subset A_3 \subset A_4 \subseteq A, a_2 \in A_2 \setminus A_0, a_3 \in A_3 \setminus A_2, a_4 \in A_4 \setminus A_3$;

(16)$A_3^* a_3 A_1^* a_1 A_0^* a_2 A_2^* a_4 A_4^*$, $A_0 \subset A_1 \subset A_2 \subset A_3 \subset A_4 \subseteq A, a_1 \in A_1 \setminus A_0, a_2 \in A_2 \setminus A_1, a_3 \in A_3 \setminus A_2, a_4 \in A_4 \setminus A_3$.

Now Lemma 7.2.10 asserts that $A^*(R_5 = S_5)$ is contained in the boolean algebra generated by the languages of the kind (1), (2), (4), (8), (16) and (3), (7), (15).

Indeed, cases (5) and (2) are the same, cases (9) and (3) are the same and cases (13), (10) and (4) are the same, too. We apply now a "reduction" procedure to the remaining cases:

$$A_3^* a_3 A_1^* a_1 A_0^* = A_3^* a_1 A_0^* \cap A_3^* a_3 A_1^*,$$

where $A_3^* a_1 A_0^*, A_3^* a_3 A_1^*$ are of kind (2);

$$A_0^* a_2 A_2^* a_4 A_4^* = A_0^* a_2 A_4^* \cap A_2^* a_4 A_4^*,$$

where $A_0^* a_2 A_4^*, A_2^* a_4 A_4^*$ are of kind (3);

$$A_1^* a_1 A_0^* a_2 A_2^* a_4 A_4^* = A_1^* a_1 A_0^* a_2 A_4^* \cap A_1^* a_1 A_2^* a_4 A_4^*,$$

where $A_1^* a_1 A_0^* a_2 A_4^*, A_1^* a_1 A_2^* a_4 A_4^*$ are of kind (4); and

$$A_3^* a_3 A_1^* a_1 A_0^* a_4 A_4^* = A_3^* a_1 A_0^* a_4 A_4^* \cap A_3^* a_3 A_1^* a_4 A_4^*,$$

where $A_3^* a_1 A_0^* a_4 A_4^*, A_3^* a_3 A_1^* a_4 A_4^*$ are of kind (4).

Now Lemma 7.2.11 asserts that $A^*(R_5 = S_5)$ is contained in the boolean algebra generated by the languages of the kind (2), (4), (8) and (16).

Indeed,

$$A_0^* = A^* \setminus \bigcup_{a \in A \setminus A_0} A^* a (A \setminus \{a\})^*$$

where $A^* a (A \setminus \{a\})^*$ is a language of kind (2).

Now, for $A_0 \neq \emptyset$

$$A_0^* a_2 A_2^* = a_2 A_2^* \cup \bigcup_{a \in A_0} A_0^* a a_2 A_2^*.$$

But

$$\begin{aligned} a_2 A_2^* &= a_2^* a_2 \cup \bigcup_{a \in A_2 \setminus \{a_2\}} a_2^* a_2 a A_2^* \\ &= a_2^* a_2 \emptyset^* \cup \bigcup_{a \in A_2 \setminus \{a_2\}} a_2^* a_2 \emptyset^* a A_2^*, \end{aligned}$$

where $a_2^* a_2 \emptyset^*$ is of kind (2) and $a_2^* a_2 \emptyset^* a A_2^*$ is of kind (4); and

$$A_0^* a a_2 A_2^* = A_0^* a \emptyset^* a_2 A_2^*$$

is of kind (4).

Also, (for $A_0 \neq \emptyset$)

$$A_3^* a_3 A_0^* a_2 A_2^* = A_3^* a_3 a_2 A_2^* \cup \bigcup_{a \in A_0} A_3^* a_3 A_0^* a a_2 A_2^*,$$

where

$$A_3^* a_3 A_0^* a a_2 A_2^* = A_3^* a_3 A_0^* a \emptyset^* a_2 A_2^*$$

is of kind (8). But

$$\begin{aligned} A_3^* a_3 a_2 A_2^* &= A_3^* a_3 a_2 a_2^* \cup \bigcup_{a \in A_2 \setminus \{a_2\}} A_3^* a_3 a_2^* a_2 a A_2^* \\ &= A_3^* a_3 a_2 a_2^* \cup \bigcup_{a \in A_2 \setminus \{a_2\}} A_3^* a_3 a_2^* a_2 \emptyset^* a A_2^* \\ &= A_3^* a_3 a_2^* a_2 \emptyset^* \cup \bigcup_{a \in A_2 \setminus \{a_2\}} A_3^* a_3 a_2^* a_2 \emptyset^* a A_2^* \end{aligned}$$

where $A_3^* a_3 a_2^* a_2 \emptyset^* a A_2^*$ is of kind (8) and $A_3^* a_3 a_2^* a_2 \emptyset^*$ is of kind (6) (thus can also be reduced).

Finally, (for $A_0 \neq \emptyset$)

$$A_3^* a_3 A_0^* a_2 A_2^* a_4 A_4^* = A_3^* a_3 a_2 A_2^* a_4 A_4^* \cup \bigcup_{a \in A_0} A_3^* a_3 A_0^* a a_2 A_2^* a_4 A_4^*,$$

where

$$A_3^* a_3 A_0^* a a_2 A_2^* a_4 A_4^* = A_3^* a_3 A_0^* a \emptyset^* a_2 A_2^* a_4 A_4^*$$

is of kind (16). But

$$A_3^* a_3 a_2 A_2^* a_4 A_4^* = A_3^* a_3 a_2 a_2^* a_4 A_4^* \cup \bigcup_{a \in A_2 \setminus \{a_2\}} A_3^* a_3 a_2^* a_2 a A_2^* a_4 A_4^*,$$

where

$$A_3^* a_3 a_2^* a_2 a A_2^* a_4 A_4^* = A_3^* a_3 a_2^* a_2 \emptyset^* a A_2^* a_4 A_4^*$$

is of kind (16) and

$$\begin{aligned} A_3^* a_3 a_2 a_2^* a_4 A_4^* &= A_3^* a_3 a_2^* a_2 a_4 A_4^* \\ &= A_3^* a_2 a_4 A_4^* \cap A_3^* a_3 a_2^* a_4 A_4^* \\ &= A_3^* a_2 \emptyset^* a_4 A_4^* \cap A_3^* a_3 a_2^* a_4 A_4^* \end{aligned}$$

where $A_3^* a_2 \emptyset^* a_4 A_4^*$, $A_3^* a_3 a_2^* a_4 A_4^*$ are of kind (4).

FIGURES

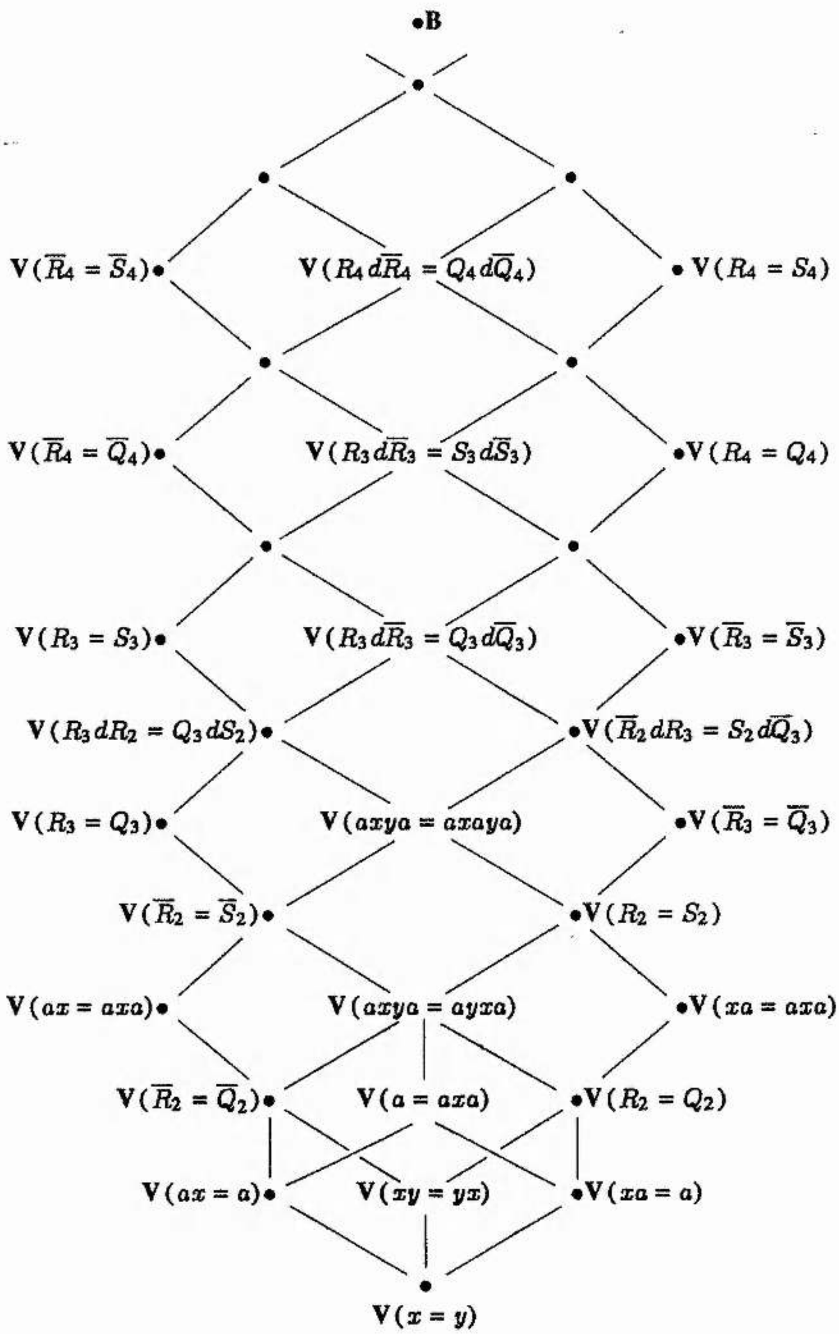


Figure 1. The lattice **LB** of varieties of bands.

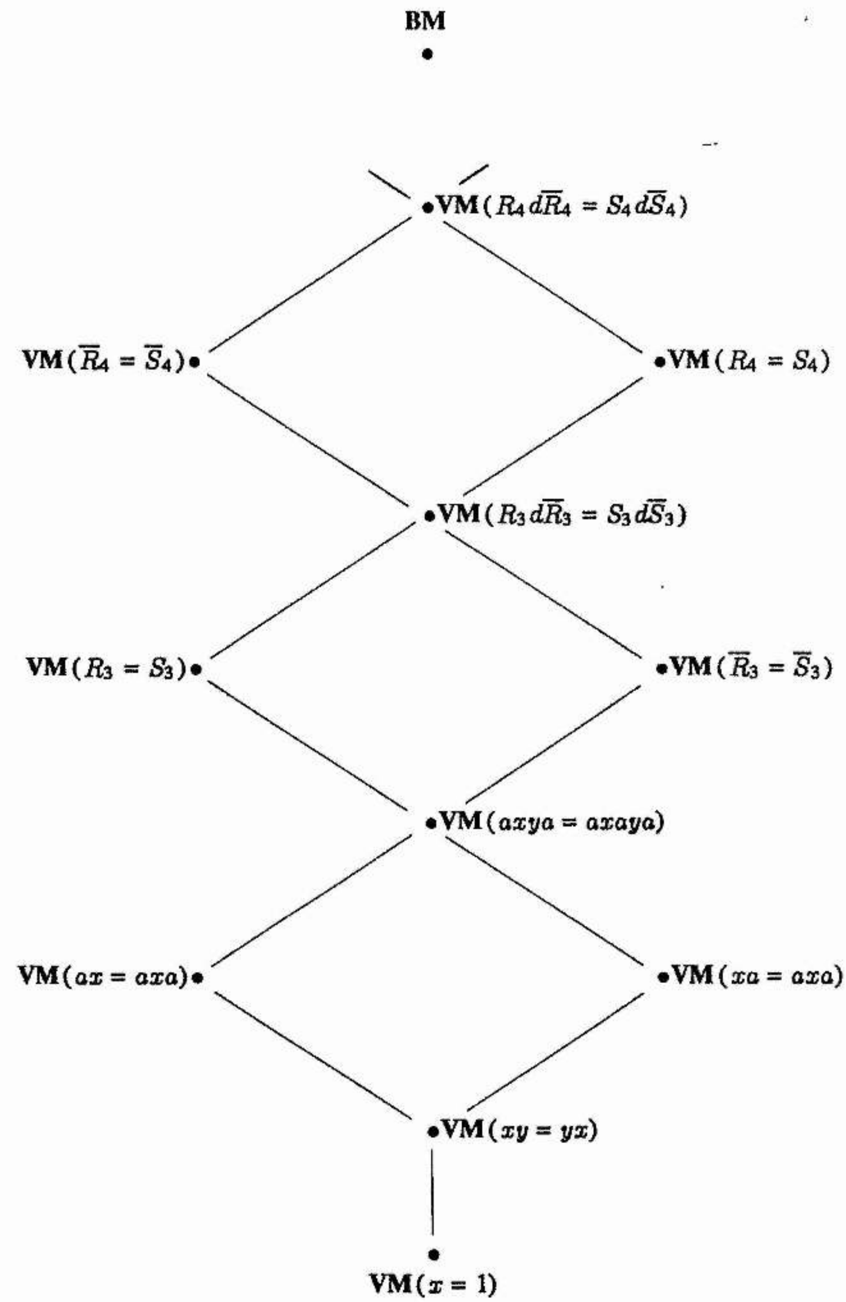


Figure 2. The lattice of varieties of band monoids.

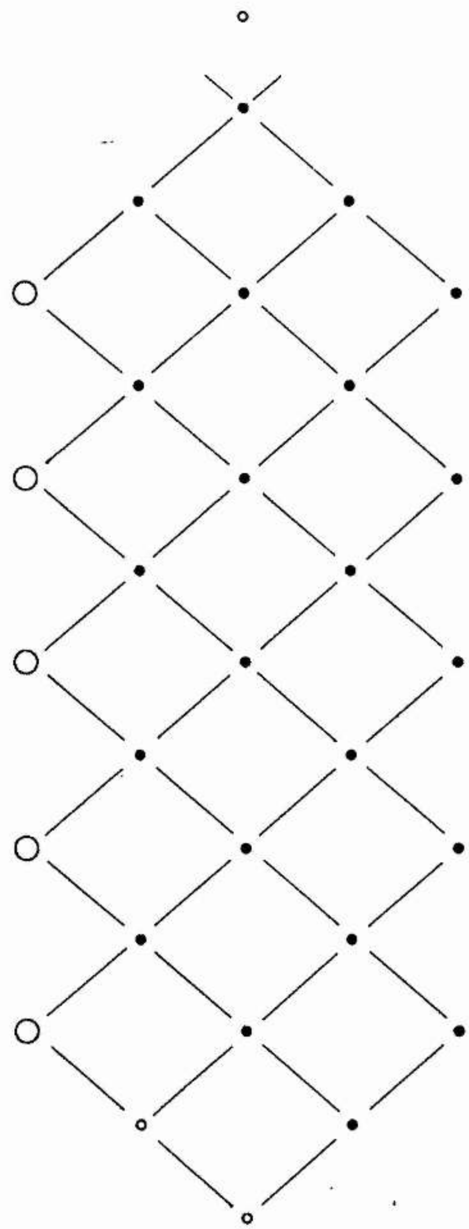


Figure 3. The lattice $\mathbf{LB}_\bullet$.

∘, ◯ – left varieties ; ◯ – strictly left varieties.

B_3 :

	a_0^3	a_1^3	a_2^3	b_1^3	b_2^3	id_3
a_0^3	a_0^3	a_0^3	a_0^3	a_0^3	a_0^3	a_0^3
a_1^3	a_1^3	a_1^3	a_1^3	a_1^3	a_1^3	a_1^3
a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3	a_2^3
b_1^3	a_0^3	a_1^3	a_1^3	b_1^3	b_2^3	b_1^3
b_2^3	a_0^3	a_1^3	a_0^3	b_1^3	b_2^3	b_2^3
id_3	a_0^3	a_1^3	a_2^3	b_1^3	b_2^3	id_3

B_4 :

	a_0^4	a_1^4	a_2^4	a_3^4	b_1^4	b_2^4	b_3^4	a_0^2	a_1^2	id_4
a_0^4	a_0^4	a_0^4	a_0^4	a_0^4	a_0^4	a_0^4	a_0^4	a_0^4	a_0^4	a_0^4
a_1^4	a_1^4	a_1^4	a_1^4	a_1^4	a_1^4	a_1^4	a_1^4	a_1^4	a_1^4	a_1^4
a_2^4	a_2^4	a_2^4	a_2^4	a_2^4	a_2^4	a_2^4	a_2^4	a_2^4	a_2^4	a_2^4
a_3^4	a_3^4	a_3^4	a_3^4	a_3^4	a_3^4	a_3^4	a_3^4	a_3^4	a_3^4	a_3^4
b_1^4	a_0^4	a_1^4	a_1^4	a_1^4	b_1^4	b_2^4	b_3^4	b_1^4	b_1^4	b_1^4
b_2^4	a_0^4	a_1^4	a_0^4	a_0^4	b_1^4	b_2^4	b_3^4	b_2^4	b_2^4	b_2^4
b_3^4	a_0^4	a_1^4	a_0^4	a_1^4	b_1^4	b_2^4	b_3^4	b_2^4	b_1^4	b_3^4
a_0^2	a_0^4	a_1^4	a_2^4	a_2^4	b_1^4	b_2^4	b_3^4	a_0^2	a_0^2	a_0^2
a_1^2	a_0^4	a_1^4	a_3^4	a_3^4	b_1^4	b_2^4	b_3^4	a_1^2	a_1^2	a_1^2
id_4	a_0^4	a_1^4	a_2^4	a_3^4	b_1^4	b_2^4	b_3^4	a_0^2	a_1^2	id_4

Figure 4. The tables of the monoids B_3 and B_4.

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